250A Homework 7

Solutions by Jaejeong Lee

Let p be prime throughout.

Question 1 Let $H \trianglelefteq G$ such that $p \nmid [G : H]$. Show that every Sylow-p-subgroup of G is contained in H.

Solution Suppose by way of contradiction that P is a Sylow p-subgroup of G which is not contained in H. Let $g \in P \setminus H$. Since g has p-power order, so does $gH \in G/H$; a contradiction to $p \nmid [G : H]$ by Lagrange's Theorem.

Question 2 Let $K \trianglelefteq G$ be a p-subgroup. Show K is contained in every Sylow-p-subgroup of G.

Solution By the second Sylow theorem, for any Sylow p-subgroup P there exists $g \in G$ such that $K \leq gPg^{-1}$. But then $K = g^{-1}Kg \leq P$, since K is normal.

Question 3 Decide upon the validity of the following statements. Give reasons (*i.e.* proofs or counterexamples) for your answers.

(i) If $H \leq G$ and S is a Sylow-p-subgroup of G, then $S \cap H$ is a Sylow-p-subgroup of H.

(ii) If S and S' are Sylow-p-subgroups of G and G', respectively, then $S \times S'$ is a Sylow-p-subgroup of $G \times G'$.

(iii) If G has a Sylow-p-subgroup of order p^n, then all subgroups of order p^{n-1} are conjugate.

Solution (i) (False) Let $G = S_3 = \langle r, s \mid r^3 = s^2 = 1, srs = r^{-1} \rangle$, $H = \langle s \rangle$, and $S = \langle rs \rangle$. Then S (and H, too) is a Sylow 2-subgroup of G but $S \cap H = \{1\}$ is not a Sylow 2-subgroup of H.
(ii) (True) Let $|G| = p^\alpha m$ and $|G'| = p^\beta n$ with $p \nmid mn$. Then $|S| = p^\alpha$ and $|S'| = p^\beta$. Since $|G \times G'| = p^{\alpha+\beta}mn$ and $|S \times S'| = p^{\alpha+\beta}$, $S \times S'$ is a Sylow p-subgroup of $G \times G'$.

(iii) (False) Let $G = S_4$, $H_1 = \{(1), (12)(34), (13)(24), (14)(23)\}$, and $H_2 = \langle(1234)\rangle$. Then $|G| = 2^3 \cdot 3$ and $|H_1| = |H_2| = 2^2$. But H_1 and H_2 are not conjugate; one is cyclic, the other not.

Question 4 If $H \leq G$ and P, P' are distinct Sylow-p-subgroups of H, show P and P' cannot be subgroups of the same Sylow-p-subgroup of G.

Solution Suppose by way of contradiction that Q is a Sylow p-subgroup of G containing both P and P'. The p-subgroup $H \cap Q$ of H contains both P and P', hence $P = H \cap Q = P'$; a contradiction.

Question 5 Recall that a simple group G has no normal subgroups save for itself and 1_G. How many simple groups are there of the following orders: 18, 56, 70, 225 and 700? Give reasons for your answer in each case.

Solution (i) $18 = 2 \cdot 3^2$; $n_3 = 1$. (ii) $56 = 2^3 \cdot 7$; $n_7 = 1$ or 8. If $n_7 = 8$, then there are 48 $(= 6 \cdot 8)$ elements of order 7. There are only 8 $(= 56 - 48)$ elements remaining, so $n_2 = 1$. (iii) $70 = 2 \cdot 5 \cdot 7$; $n_7 = 1$. (iv) $225 = 3^2 \cdot 5^2$; $n_5 = 1$. (v) $700 = 2^2 \cdot 5^2 \cdot 7$; $n_5 = 1$.

Question 6 Suppose p and q are distinct primes. Show there are no simple groups of order p^2q.

Solution [p144, Dummit & Foote]