250A Homework 8

Solutions by Jaejeong Lee

Question 1 Comment on the (historical) significance of the solvability of S_5.

Solution (cf. Section 14.7 [Dummit & Foote])

Question 2 Compute the derived series for dihedral groups D_{2n} and their quotients. Are these groups solvable? Of what length?

Solution Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle$ and observe $[r^k s^i, r^l s^j] = r^{2(k-l)}$. We have

$$D_{2n}^{(1)} = \begin{cases} \langle r \rangle \simeq \mathbb{Z}_n & \text{if } n \text{ is odd} \\ \langle r^2 \rangle \simeq \mathbb{Z}_{n/2} & \text{if } n \text{ is even} \end{cases}$$

and $D_{2n}^{(2)} = 1$, hence D_{2n} is solvable of length 2. Finally, note that

$$D_{2n}/D_{2n}^{(1)} \simeq \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is odd} \\ V_4 & \text{if } n \text{ is even.} \end{cases}$$

Question 3 As in question 2 but upper triangular invertible $n \times n$ matrices.

Solution Let T_n be the group of upper triangular invertible $n \times n$ matrices and let

$$T_{n,k} = \{I + A \in T_n \mid A_{ij} = 0 \text{ for } j - i \leq k\}.$$

Using induction on n, one can verify that $T_n^{(k)} = T_{n,k-1}$ for each $k \geq 1$. Since $T_n^{(n)} = T_{n,n-1} = \{I\}$, T_n is solvable of length n. Note that $T_n/T_n^{(1)} \simeq (\mathbb{R}^\times)^n$ and $T_n^{(k)}/T_n^{(k+1)} \simeq \mathbb{R}^{n-k}$ for $k \geq 1$.[1]

Question 4 Find an example of both a finite and infinite group with a quotient group not isomorphic to any subgroup.

[1] You may want to take MAT261 this winter.

Solution (cf. Section 5.5 [p181, Dummit & Foote]) (i) Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Note that $Q_8/\langle -1 \rangle \simeq V_4$ but every order 4 subgroup of Q_8 is cyclic. (ii) $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$ and every nontrivial subgroup of $\mathbb{Z}$ is infinite.

Question 5 Let p be prime and F_p the field with p elements[2].

(i) Let $N \leq Gl(3, F_p)$ be the group of 3×3 F_p-valued upper triangular matrices with 1's on the diagonal. Show that N is non-abelian of order p^3.

(ii) Compute the order of $Gl(3, F_2)$. Identify the Sylow-p-subgroups and decide if any of them are normal.

Solution (i) It is clear that $|N| = p^3$. Since

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

N is non-abelian.
(ii) In general, we have

$$|Gl(n, F_p)| = (p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1}).$$

(cf. [pp412-413, Dummit & Foote]) Thus, $|Gl(3, F_2)| = (2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 168 = 2^3 \cdot 3 \cdot 7$. For the rest, see [pp207-212, Dummit & Foote].

[2]Take integers mod p with multiplication and addition defined in the usual way.