250A Homework 9

Solutions by Jaejeong Lee

Question 1 (DF 7.1.15) Rings R where $a^2 = a \; \forall \, a \in R$ are called Boolean. Prove that Boolean rings are abelian and $a + a = 0 \; \forall \, a \in R$.

Solution For all $a, b \in R$, we have $a+b = (a+b)^2 = a^2+b^2+ab+ba = a+b+ab+ba$, hence $ab+ba = 0$. Setting $b = a$, we get $0 = a^2+a^2 = a+a$. Therefore, each element of R is its own additive inverse. In particular, $ab = -ab = ba$, so R is commutative.

Question 2 Let R be a commutative ring and call $x \in R$ nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$. Prove that the set of nilpotent elements is an ideal I (i.e. an additive subgroup closed under left and right multiplication). Show that the quotient R/I has no non-trivial nilpotent elements. Give an example of a ring in which the set of nilpotent elements is not an ideal.

Solution (cf. Exercises 29-31 [p250, Dummit & Foote])
(i) Let $I \subset R$ be the set of nilpotent elements, $a, b \in I$ and $r \in R$. Then $a^m = b^n = 0$ for some $m, n \in \mathbb{N}$. Since R is commutative, we see $(a-b)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} a^k (-b)^{m+n-k} = 0$, thus I is an additive subgroup of R. Moreover, since $(ra)^m = r^m a^m = 0$ and $(ar)^m = a^m r^m = 0$, I is an ideal of R.
(ii) Suppose $(a+I)^k = I \in R/I$ for some $k \in \mathbb{N}$. Since $I = (a+I)^k = a^k + I$, we see $a^k \in I$. Thus $(a^k)^l = 0$ for some $l \in \mathbb{N}$, hence $a \in I$ and $a + I = I$.
(iii) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z})$ are nilpotent because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. But $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not nilpotent because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, the set of nilpotent elements in $M_{2\times 2}(\mathbb{Z})$ is not closed under addition.

Question 3 Let R be a ring and $f, g : \mathbb{Q} \to R$ be ring homomorphisms coinciding on the integers, i.e. $f|_{\mathbb{Z}} = g|_{\mathbb{Z}}$. Show that $f = g$.

Solution For all $\frac{m}{n} \in \mathbb{Q}$, we have

$$\begin{aligned} f(\frac{m}{n}) &= f(m \cdot \frac{1}{n}) = f(m)f(\frac{1}{n}) \\ &= g(m)f(\frac{1}{n}) = g(\frac{m}{n} \cdot n)f(\frac{1}{n}) = g(\frac{m}{n})g(n)f(\frac{1}{n}) \\ &= g(\frac{m}{n})f(n)f(\frac{1}{n}) = g(\frac{m}{n})f(n \cdot \frac{1}{n}) = g(\frac{m}{n})f(1) \\ &= g(\frac{m}{n})g(1) = g(\frac{m}{n} \cdot 1) = g(\frac{m}{n}). \end{aligned}$$

Question 4 Give examples of rings R where (i) R has a left ideal that is not a right ideal (ii) R has zero divisors but R/I does not for some ideal I.

Solution (i) See Examples (8) [p245, Dummit & Foote].
(ii)[1] $\mathbb{Z}_4$ has a zero divisor, namely, $2 \cdot 2 = 0$, but $\mathbb{Z}_4/(2) \simeq \mathbb{Z}_2$ does not.

[1]See Proposition 11, Proposition 13 and Corollary 14 [pp254-256, Dummit & Foote]: Every ring with identity has a maximal ideal, hence a prime ideal, the quotient over which being an integral domain.