Homework 3
Solutions

1. Certainly $\mathrm{SL}_n(\mathbb{R})$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$: it is closed under matrix multiplication since $\det AB = \det A \det B = 1$ if $A, B \in \mathrm{SL}_n(\mathbb{R})$, the inverse of $A \in \mathrm{SL}_n(\mathbb{R})$ is in $\mathrm{SL}_n(\mathbb{R})$ since $\det A^{-1} = (\det A)^{-1} = 1$ and the identity matrix is also in $\mathrm{SL}_n(\mathbb{R})$.

   Now let $S \in \mathrm{SL}_n(\mathbb{R})$ and $G \in \mathrm{GL}_n(\mathbb{R})$. Then $\det GSG^{-1} = \det G (\det G)^{-1} = 1$ so that $GSG^{-1} \in \mathrm{SL}_n(\mathbb{R})$ for all $G \in \mathrm{GL}_n(\mathbb{R})$. Hence $G\mathrm{SL}_n(\mathbb{R})G^{-1} = \mathrm{SL}_n(\mathbb{R})$ which shows that $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.
2. The center of $\mathrm{GL}_2(\mathbb{R})$ is
$$Z(\mathrm{GL}_2(\mathbb{R})) = \{\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{R}\backslash\{0\}\}.$$
3. All elements in $G/Z(G)$ are of the form $gZ(G)$ where $g \in G$. Suppose that $G/Z(G)$ is cyclic of order n. Then there is an $x \notin Z(G)$ such that
$$(xZ(G))^n = x^n Z(G) = Z(G).$$
   This implies that $x^n \in Z(G)$. Now $G/Z(G) = \langle xZ(G) \rangle = \{x^i Z(G) \mid i = 0, 1, \ldots, n-1\}$. Hence every $g \in G$ can be expressed as $g = x^i z$ where $z \in Z(G)$. Let $h = x^j z'$ be another element in G. Then
$$gh = x^i z x^j z' = z x^i x^j z' = z x^j x^i z' = x^j z' x^i z = hg$$
   since z, z' are in the center of G. This shows that any two elements $g, h \in G$ commute and hence G is abelian.
4. Let $A = \{aH \mid a \in G\}$ be the set of all left cosets of H. Then $|A| = n$. G acts on A as $g \cdot (aH) = (ga)H$. We showed in class that every left group action of G and A is in bijective correspondence to a homomorphism $\varphi : G \to S_A$ given by $g \mapsto \sigma_g$ where $\sigma_g : A \to A$ with $aH \mapsto (ga)H$. The kernel of φ is a normal subgroup of G. Hence the index of $\ker\varphi$ is $[G : \ker\varphi] \leq n!$ which is the order of S_A.
5. First we prove the forward direction. Take $\bar{x}, \bar{y} \in \bar{G}$. Then $\bar{x} = xN$ and $\bar{y} = yN$ for some $x, y \in G$. If $\bar{x}\bar{y} = \bar{y}\bar{x}$ then $xyN = yxN$ or equivalently $x^{-1}y^{-1}xyN = N$ which implies that $x^{-1}y^{-1}xy \in N$.

For the reverse direction note that all steps are also true in the reverse direction.

6. Let G be a group and N a subgroup of index 2. If $a \in N$ then $aN = Na$. If $a \notin N$ then $aN \neq N$ since N has index 2. Similarly $Na \neq N$. Since N has index 2 this implies that $aN = Na$ so that $aNa^{-1} = N$. Hence for all $a \in G$ we have $aNa^{-1} = N$ so that N is a normal subgroup.
7. The only candidates for normal subgroups of S_4 are conjugacy classes or unions thereof. By the extra problem on Problem set 2 we know that all conjugacy classes of S_n can be described by partitions of n. The partitions of 4 are $(1,1,1,1), (2,1,1), (2,2), (3,1), (4)$. Certainly $\{1\}$ and S_4 are normal subgroups of S_4. The set of odd permutations cannot form a subgroup since the composition of two odd permutations is an even permutation. Since the 4-cycles and 2-cycles are odd they cannot form a subgroup. The set of 3-cycles does not form a subgroup since for example $(123)(124) = (13)(24)$ which is not a 3-cycle. Hence the only candidates for non-trivial normal subgroups are the alternating group A_4 of permutations which can be written by an even number of transpositions and $K = \{1, (12)(34), (13)(24), (14)(23)\}$. Note that K is isomorphic to the Klein 4-group. K is a normal subgroup since for all $\sigma \in S_4$ we have $\sigma(ab)(cd)\sigma^{-1} = (\sigma(a)\sigma(b))(\sigma(c)\sigma(d)) \in K$ where all a, b, c, d are distinct and $\sigma 1 \sigma^{-1} = 1 \in K$. One can check that A_4 is a subgroup of S_4 and has order 12. Since hence the index of A_4 in S_4 is 2 it follows by Problem 6 that A_4 is normal.
8. Set $T = \{g \in G \mid |g| < \infty\}$. Then T is nonempty since $1 \in T$. Let $g, h \in T$. Then there exist integers n, m such that $g^n = h^m = 1$. Hence $(gh^{-1})^{nm} = g^{nm}h^{-nm} = 1$ since G is abelian. This shows that T is a subgroup of G.

   Take the non-abelian group $G = \mathrm{GL}_2(\mathbb{R})$ and
   $$x = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$
   Both elements have order 6, but $xy = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has infinite order. Hence in this case T is not a subgroup of G.
9. (a) The elements in $\mathbb{Q}/\mathbb{Z}$ are of the form $a + \mathbb{Z}$. Hence the rational number satisfying $0 \leq a < 1$ give all the representatives.

   (b) Take $a \in \mathbb{Q}$ with $0 \leq a < 1$. Then $a = \frac{n}{m}$ for some $n, m \in \mathbb{Z}$, $n \geq 0, m > 0$ and $\gcd(n, m) = 1$. Hence $ma + \mathbb{Z} = \mathbb{Z}$ so that a has order m. Since m can be arbitrarily large, the order can be arbitrarily large.

(c) No $a+\mathbb{Z}$ has finite order if a is irrational since am is never an integer for any positive integer m. By part (b) all elements in $\mathbb{Q}/\mathbb{Z}$ have finite order. Hence the torsion group of $\mathbb{R}/\mathbb{Z}$ is $\mathbb{Q}/\mathbb{Z}$.

(d) Define the map $\varphi : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$ given by $a + \mathbb{Z} \mapsto e^{2\pi i a}$. This map is well-defined since $e^{2\pi i m} = 1$ for all $m \in \mathbb{Z}$. It is a homomorphism since $\varphi((a+\mathbb{Z}) + (b+\mathbb{Z})) = \varphi((a+b)+\mathbb{Z}) = e^{2\pi i(a+b)} = e^{2\pi i a}e^{2\pi i b} = \varphi(a+\mathbb{Z})\varphi(b+\mathbb{Z})$. It is easy to see that φ is injective and surjective.