Homework 5
Solutions

(1) (a) Suppose H char G. In particular, $\varphi_g : G \to G$ mapping $a \mapsto gag^{-1}$ is in $\mathrm{Aut}(G)$. By assumption $\varphi_g(H) = H$ so that $gHg^{-1} = H$ which means that H is normal in G.
(b) Let $g \in G$. Since $K \trianglelefteq G$, the map $\varphi_g : K \to K$ mapping $k \mapsto gkg^{-1}$ is in $\mathrm{Aut}(K)$. Since H char K this means that $\varphi_g(H) = gHg^{-1} = H$ for all $g \in G$ so that H is normal in G.

(2) (a) The set of inner automorphisms $\mathrm{Inn}(G)$ is the set of all maps $\varphi_g : G \to G$ of the form $\varphi_g(x) = gxg^{-1}$. Let $\tau \in \mathrm{Aut}(G)$ and $g \in G$. Then for all $x \in G$

$$(\tau\varphi_g\tau^{-1})(x) = \tau(\varphi_g(\tau^{-1}(x))) = \tau(g\tau^{-1}(x)g^{-1})$$
$$= \tau(g)x\tau(g)^{-1} = \varphi_{\tau(g)}(x).$$

Hence $\tau\varphi_g\tau^{-1} \in \mathrm{Inn}(G)$ and $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.
(b) An infinite cyclic group is isomorphic to the additive group $\mathbb{Z}$, and a finite cyclic group of order n is isomorphic to Z_n. Both $\mathbb{Z}$ and Z_n are abelian, hence $\mathrm{Inn}(G) = \{1\}$ and $\mathrm{Out}(G) \cong \mathrm{Aut}(G)$. If G is a finite cyclic group we showed before (Dummit, Foote Section 2.3 Ex. 26) that $\mathrm{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ which has order $\varphi(n)$.

(3) Note that H is a Sylow 5-subgroup and the number of Sylow 5-subgroups is of the form $5k+1$ and divides 20. Therefore H is the unique Sylow 5-group and hence $H \trianglelefteq G$.

(4) Let $\{1\} \neq H \trianglelefteq G$ where G is a p-group. Then G acts on H by conjugation and the set of fixed points is

$$\ker H = \{h \in H \mid ghg^{-1} = h \quad \forall g \in G\} = H \cap Z(G).$$

But G is a p-group so that by the magic lemma $|\ker H| \equiv |H| \equiv 0 \mod p$ so that p divides $|\ker H|$. This implies in particular, $H \cap Z(G) \neq \{1\}$.

(5) Let G be a finite group and let H be a normal p-subgroup of G. Then H is contained in some Sylow p-subgroup K by the first Sylow theorem. Moreoever, for every $g \in G$

$$H = gHg^{-1} \leq gKg^{-1}$$

so that H is contained in every Sylow p-subgroup by the second Sylow theorem.

(6) Let G be a simple group of order $168 = 7 \times 24$. If m is the number if Sylow 7-subgroups, then $m > 1$, $m = 7k + 1$ and m divides 24. It follows that $m = 8$ so that there are $6 \times 8 = 48$ elements of order 7 in G.