**Homework 6**
Solutions

(1) (a) This follows from the fact that every $\langle x_i \rangle$ is abelian.
(b)

$$\text{im}\varphi = \langle x_1^p \rangle \times \cdots \times \langle x_n^p \rangle$$
$$\text{ker}\varphi = \langle x_1^{p^{\alpha_1 - 1}} \rangle \times \cdots \times \langle x_n^{p^{\alpha_n - 1}} \rangle.$$

(c) From (b) it is clear that both ker$\varphi$ and $A/\text{im}\varphi$ are isomorphic to $Z_p \times \cdots \times Z_p = E_{p^n}$ and hence the rank is $n$.

(2) Since $H, K$ char $G$ we have in particular that $H, K \trianglelefteq G$. Hence by Theorem 9 on page 173 $G \cong H \times K$ and the map $\varphi : G \to H \times K$ with $hk \mapsto (h, k)$ is a well-defined isomorphism. Now define

$$\Phi : \text{Aut}(G) \to \text{Aut}(H) \times \text{Aut}(K)$$
$$\tau \mapsto \varphi \circ \tau \circ \varphi^{-1} = (\tau, \tau).$$

This is well-defined since $H, K$ char $G$. Also $\Phi$ is a homomorphism since for $(h, k) \in H \times K$ and $\tau, \sigma \in \text{Aut}(G)$ we have

$$\Phi(\tau \circ \sigma)(h, k) = (\tau \circ \sigma(h), \tau \circ \sigma(k))$$
$$= (\tau, \tau) \circ (\sigma, \sigma)(h, k) = \Phi(\tau) \circ \Phi(\sigma)(h, k).$$

To show injectivity, assume that $\tau, \sigma \in \text{Aut}(G)$ such that $\tau \neq \sigma$. This means that there is a $g \in G$ such that $\tau(g) \neq \sigma(g)$. But every $g \in G$ can be written uniquely as $g = hk$ for some $h \in H$ and $k \in K$. Hence $\tau(h)\tau(k) \neq \sigma(h)\sigma(k)$ implies that either $\tau(h) \neq \sigma(h)$ or $\tau(k) \neq \tau(k)$. Hence $\Phi(\tau) \neq \Phi(\sigma)$. To show surjectivity, let $\alpha \in \text{Aut}(H)$ and $\beta \in \text{Aut}(K)$. Define $\tau : G \to G$ by $\tau(g) = \alpha(h)\beta(k)$ if $g = hk$. To see that $\tau \in \text{Aut}(G)$ one needs to use that $hk = kh$ for all $h \in H$, $k \in K$.

Finally, if $G$ is abelian then every subgroup is normal. Hence, if $|G| = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, there is a unique Sylow $p_i$-subgroup for every $1 \leq i \leq t$. By Corollary 20 on page 144 all Sylow $p_i$-subgroups are therefore characteristic and by Lagrange only intersect in the identity. The assertion that Aut$(G)$ is isomorphic to the direct product of the automorphism groups of its Sylow subgroups then follows from the above arguments.

(3) By Homework 4 #5 we know that $G/(A \cap B) \cong G/A \times G/B$ (the same arguments work since $AB$ is a subgroup of $G$). If both $G/A$ and $G/B$ are abelian then so is $G/(A \cap B)$.

(4) If $G$ is the direct product of $H$ and $K$, then the map $\mu : H \times K \to G$ defined by $\mu(h,k) = hk$ is an isomorphism, so that every element in $G$ can be written uniquely as a product $hk$. Let $\pi_H : H \times K \to H$ with $\pi_H(h,k) = h$ be the canonical projection so that $p_H = \pi_H \mu^{-1} : G \to H$ is the epimorphism $hk \mapsto h$. It follows that $\ker p_H = K$ so that $K \trianglelefteq G$ and $G/K \cong H$ by the first isomorphism theorem. Reversing the roles of $K$ and $H$, we see tht $\ker p_K = H$ so that $H \trianglelefteq G$ and $G/H \cong K$.

(5) Let $G$ be a group of order $n$.

**Case** $n = 12$**.** We may assume that $G$ has a Sylow 3-subgroup that is not normal. Therefore the number of such Sylow 3-subgroups must be 4 by the third Sylow theorem and consequently $G$ has exactly $8 = 4 \times 2$ elements of order 3. The remaining 4 elements form a unique and hence normal Sylow 2-subgroup.

**Case** $n = 28$**.** The third Sylow theorem implies that the number of Sylow 7-subgroups is 1 and is therefore normal.

**Case** $n = 56$**.** We may assume that $G$ has a Sylow 7-subgroup that is not normal. By the third Sylow theorem the number of Sylow 7-subgroups is 8 and consequently $G$ has exactly $48 = 8 \times 6$ elements of order 7. The remaining 8 elements must form a unique and hence normal Sylow 2-subgroup.

**Case** $n = 200$**.** The third Sylow theorem implies that the number of Sylow 5-subgroups is 1 and hence is normal.

(6) $G$ acts on $H$ by conjugation since $H$ is normal. $\mathrm{Aut}(H)$ has 2 elements, so $G$ can only act trivially by Lagrange's theorem. Hence $H$ is central since if $g \in G$ and $h \in H$ then $ghg^{-1} = h$.

Assuming that $|H|$ is prime so that we know exactly what $\mathrm{Aut}(H)$ is, the argument works as long as $|\mathrm{Aut}(H)|$ is a power of 2. In other words, $|H|$ is of the form $2^k + 1$. The primes of this form are called the Fermat primes and the only ones known are 3,5,17,257, and 65537. It is conjectured that there are no others.