Homework 7
Solutions

1. Let G be a group with $|G| = 28$. Let P be a Sylow 7-subgroup of G and K a Sylow 2-subgroup of G. The number of Sylow 7-subgroups of G is of the form $n_7 = 1 + 7k$ and has to divide 4. Hence $n_7 = 1$ and P is normal in G. By Lagrange $K \cap P = 1$. Hence G is the semidirect product of P and K with respect to some homomorphism $\varphi : K \to \mathrm{Aut}(P)$. Note that $P \cong Z_7$, $K \cong Z_4$ or $Z_2 \times Z_2$, and $\mathrm{Aut}(P) \cong Z_6 \cong Z_2 \times Z_3$.
   Case 1. Suppose $K = Z_4 = \langle x \rangle$. Let φ_1 be the trivial homomorphism which induces the action of K on P by $x \cdot p = p$. In this case $G \cong Z_7 \times Z_4 \cong Z_{28}$. Let φ_2 be the homomorphism which induces the action $x \cdot p = p^{-1}$. This group is nonabelian since for example $(1, x)(p, x^2) = (p^{-1}, x^3)$ and $(p, x^2)(1, x) = (p, x^3)$. Also, $(1, x)$ has order 4 so that $G = P \times_{\varphi_2} K$ is not isomorphic to D_{28} which does not have an element of order 4.
   Case 2. Suppose $K = Z_2 \times Z_2 \cong \langle x \rangle \times \langle y \rangle$. Let φ_3 be the trivial homomorphism. Then $G \cong Z_7 \times Z_2 \times Z_2 \cong Z_{14} \times Z_2$. Let φ_4 be the homomorphism which induces the action $x \cdot p = p^{-1}$ and $y \cdot p = p^{-1}$. Then (p, xy) has order 14, $(1, x)$ has order 2, and $(1, x)(p, xy) = (p^{-1}, xy)(1, x)$ so that $G \cong D_{28}$. Let φ_5 be the homomorphism which induces the action $x \cdot p = p^{-1}$ and $y \cdot p = p$. Then $(1, y)$ is in the center, (p, x) has order 2, $(p, 1)$ has order 7 and $(p, x)(p, 1) = (p^{-1}, 1)(p, x)$. Hence in this case $G \cong Z_2 \times D_{14} \cong D_{28}$.
   Altogether there are four distinct groups of order 28, namely Z_{28}, $Z_2 \times Z_{14}$, D_{28}, and $Z_7 \times_{\varphi_2} Z_4$.
2. Let G_1 be a maximal normal subgroup of G (that is, there is no normal subgroup $H \trianglelefteq G$ which contains G_1). Then G/G_1 is simple by the fourth isomorphism theorem. Let G_2 be a maximal normal subgroup of G_1, and so on. Since G is finite, this process terminates for some $G_n = 1$. Hence $1 = G_n \leq G_{n-1} \leq \cdots \leq G_1 \leq G$ is a composition series.
3. If N_{i+1}/N_i is abelian and $N_i \trianglelefteq N \trianglelefteq N_{i+1}$, then N/N_i is abelian since it is a subgroup of N_{i+1}/N_i. Furthermore N_{i+1}/N is abelian

since it is isomorphic to the quotient $(N_{i+1}/N_i)/(N/N_i)$ by the third isomorphism theorem.

4. If $N_i \trianglelefteq N \trianglelefteq N_{i+1}$ with $N_i \neq N$ and $N \neq N_{i+1}$, then N/N_i is a proper normal subgroup of N_{i+1}/N_i and every normal subgroup of N_{i+1}/N_i has this form by the fourth isomorphism theorem. The conclusion follows from the observation that a subnormal series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = G$ has a proper refinement if and only if there is a subgroup N such that $N_i \trianglelefteq N \trianglelefteq N_{i+1}$ with $N_i \neq N$ and $N \neq N_{i+1}$ for some i.
5. Let $|G| = p^2q$. We claim that G has a normal Sylow subgroup H.
   Case 1. Let $p > q$. In this case the number of Sylow p-subgroups is $n_p = 1$ and hence there is a normal Sylow p-subgroup.
   Case 2. Let $q > p$. The number of Sylow q-subgroups is of the form $kq+1$ and divides p^2. If $n_q = 1$ we are done. If $n_q \neq 1$, then $q > p$ forces $kq + 1 = p^2$. Therefore there are exactly $(q-1)p^2 = p^2q - p^2$ elements of order q in G. The remaining p^2 elements must comprise the unique Sylow p-subgroup of G.

   Now this shows that there is a normal Sylow subgroup of order p^2 or q. Hence G/H has order q or p^2. In either case, both H and G/H are abelian (see HW 4, Problem 1) and hence solvable so that G is solvable.
6. Let G be a nilpotent group and $1 \neq N \trianglelefteq G$. Then for some Sylow p-subgroup P of G, we must have $N \cap P \neq 1$. Therefore by HW 5 Problem 4, $N \cap Z(P) \neq 1$. But since G is the direct product of its Sylow subgroups, $Z(P) \leq Z(G)$ so that $N \cap Z(G) \neq 1$.