The Bernoulli Numbers

Here is a fun problem that relies on eigenvalues and eigenvectors for its solution:

Captain Conundrum gives Queen Quandary a pair of newborn doves, male and female for her birthday. After one year, this pair of doves breed and produce a pair of dove eggs. One year later these eggs hatch yielding a new pair of doves while the original pair of doves breed again and an additional pair of eggs are laid. Captain Conundrum is very happy because now he will never need to buy the Queen a present ever again!

Let us say that in year zero, the Queen has no doves. In year one she has one pair of doves, in year two she has two pairs of doves *etc...* Call F_n the number of pairs of doves in years n. For example, $F_0 = 0$, $F_1 = 1$ and $F_2 = 1$. Assume no doves die and that the same breeding pattern continues well into the future. Then $F_3 = 2$ because the eggs laid by the first pair of doves in year two hatch. Notice also that in year three, two pairs of eggs are laid (by the first and second pair of doves). Thus $F_4 = 3$.

(1) Compute F_5 and F_6.

(2) Explain why (for any $n \geq 2$) the following *recursion relation* holds

$$F_n = F_{n-1} + F_{n-2}\,.$$

(3) Let us introduce a column vector $X_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$. Compute X_1 and X_2. Verify that these vectors obey the relationship

$$X_2 = MX_1 \text{ where } M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

(4) Show that $X_{n+1} = MX_n$.

(5) Diagonalize M. (*I.e.*, write M as a product $M = PDP^{-1}$ where D is diagonal.)

(6) Find a simple expression for M^n in terms of P, D and P^{-1}.

(7) Show that $X_{n+1} = M^n X_1$.

(8) The number

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is called the *golden ratio*. Write the eigenvalues of M in terms of φ.

(9) Put your results from parts (c), (f) and (g) together (along with a short matrix computation) to find the formula for the number of doves F_n in year n expressed in terms of φ, $1-\varphi$ and n.