Change of Basis Worksheet

By now you should understand that matrices are a great tool for computations involving linear transformations between finite dimensional vector spaces. However matrix elements change depending which basis you use for your problem. Therefore it is crucial to understand what happens to matrices when you change basis.

Let $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ be bases for a vector space V. Consider a linear transformation $p \in \hom(V, V)$ defined by

$$p(v_1) = w_1\,, \quad p(v_2) = w_2\,, \quad \dots \quad p(v_n) = w_n\,.$$

Answer the following questions (with justifications):

(o) Why was is sufficient to specify only $p(v_1), \dots, p(v_n)$ to completely determine the map p?

(i) What is the matrix of p using the basis $(v_1, \dots, v_n)$ for the inputs and $(w_1, \dots, w_n)$ for the outputs of p?

(ii) What is $\ker(p)$?

(iii) What is $\operatorname{im}(p)$?

(iv) Is p invertible?

We are going to use the linear transformation p to change between the bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$. Hopefully you found a simple answer for part (i). More interesting would be to calculate the matrix of p using the same basis for inputs and outputs. Lets try an example. Let

$$v_1 = (1, 0, 0)\,, \quad v_2 = (0, 1, 0)\,, \quad v_3 = (0, 0, 1) \tag{1}$$

and

$$w_1 = (1, 1, 0)\,, \quad w_2 = (0, 1, 1)\,, \quad w_3 = (1, 0, 1). \tag{2}$$

Now compute the following matrices:

(i) The matrix of p using the basis $(v_1, \dots, v_n)$ for both the inputs and outputs.

(ii) The matrix of p using the basis $(w_1, \dots, w_n)$ for both the inputs and outputs.

Compare the two matrices and *comment* on what you find.

Lets consider the general case, suppose the basis $(w_1, \dots, w_n)$. Suppose that

$$w_i = \sum_{j=1}^{n} a_{ji} v_j\,.$$

Compute the matrix of p in the basis $(v_1, \dots, v_n)$ (for both inputs and outputs). Note that $p(w_i) = \sum_{j=1}^{n} a_{ji} p(v_j) = \sum_{j=1}^{n} a_{ji} w_j$ which explains part (ii) above. Now *answer* the following crucial question: *Does the matrix of p have an inverse?*

Let p^{-1} be the inverse of the linear transformation p. *Compute* the following vectors:

$$p^{-1}(w_1)\,, p^{-1}(w_2)\,, \dots\,, p^{-1}(w_n)\,.$$

Now *explain* how to compute the matrix of p^{-1} in the basis $(v_1, \dots, v_n)$ for both the inputs and outputs. Put your explanation to work by *computing* the matrix of p^{-1} in the basis (v_1, v_2, v_3) in equation (1) where $w_i = p(v_i)$ is given by equation (2). (Don't use Gaussian elimination or Cramer's rule on the matrix of p that you found previously but rather focus on expressing the v's as linear combinations of the w's.)

Our last step is to suppose we have a linear transformation $f : V \to V$ whose matrix in the basis $(v_1, \dots, v_n)$ is M. Also imagine that we have computed the matrices of p and p^{-1} in the basis $(v_1, \dots, v_n)$ and found matrices P and P^{-1}, respectively. We want to find a formula for the matrix of f in the basis $(w_1, \dots, w_n)$. So far we have shown that

$$\big(w_1, \dots, w_n\big) = \big(v_1, \dots, v_n\big)P\,.$$

Use this to *find a formula* for the matrix M' in terms of P, M and P^{-1} where

$$\big(f(w_1), \dots, f(w_n)\big) = \big(w_1, \dots, w_n\big) = \big(v_1, \dots, v_n\big)M'\,.$$

Finally, return to the example of equation (1) and take

$$f(v_1) = 2v_1 - \frac{1}{2}v_2 + \frac{1}{2}v_3\,, \quad f(v_2) = -v_1 + \frac{3}{2}v_2 - \frac{1}{2}v_3\,, \quad f(v_3) = v_1 + \frac{1}{2}v_2 + \frac{5}{2}v_3\,.$$

Compute the matrix M of f in the basis of equation (1) and then *recompute* it in the basis of equation (2) using your change of basis formula for M'.