Fields

The aim is to teach yourself the basic notions of fields. A field $\mathbb{F} := (F, \oplus, \odot)$ is a set F whose elements can be thought of as numbers as well as rules $\oplus$ and $\odot$ for "adding" and "multiplying" them. The set F must contain special elements $0, 1 \in F$ that play the *rôle* of zero and one. If $\mathbb{F}$ is a finite field (F has finitely many elements), we can encode the rules for addition and multiplication in tabular form. The simplest example of a field $\mathbb{Z}_2 = \{0, 1\}$ has only a zero and unit element (often called "bits") with addition and multiplication tables:

$\oplus$	0	1		$\odot$	0	1
0	0	1		0	0	0
1	1	0		1	0	1

These are often called addition and multiplication modulo two, since the usual $1 + 1 = 2$ is replaced by $1 \oplus 1 = 0$.

Here are some exercises for you to try:

(1) Find out the definition of a field. For each rule, give an example of how it works for ordinary real numbers.

(2) Check that the rules given above for $\mathbb{Z}_2$ obey all the field axioms.

(3) Give the addition and multiplication tables for $\mathbb{Z}_3$ (*i.e.* addition and multiplication modulo 3). Do the same for $\mathbb{Z}_4$. Which of these is a field. Why?

(4) Find out what complex numbers and quaternions are. Explain how to add and multiply them. Which of these is a field and why?