Matrix Worksheet

You have now seen linear transformations. Also many of you have seen matrices. For finite dimensional vector spaces these two notions are very closely related. In fact we saw on the previous worksheet that a column vector was a way of specifying a vector given a basis. Matrices are a way of specifying a linear transformation given bases. Lets consider a linear transformation

$$L : V \longrightarrow W\,,$$

where $\dim V = n$ and $\dim W = m$. Moreover suppose $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ are bases for V and W.

Given $L(v_1) = u_1, L(v_2) = u_2, \ldots, L(v_n) = u_n$, compute $L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$. Explain why knowing what L does to basis vectors v_i completely determines L.

To specify the vectors $L(v_1), \ldots, L(v_n) \in W$ we can express them in the basis $(w_1, \ldots, w_m)$

$$\begin{aligned}
L(v_1) &= a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\
L(v_2) &= a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\
&\vdots \\
L(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m
\end{aligned}$$

The coefficients a_{ij} constitute the matrix of L. To make this easier to handle, remember that we wrote the column vector of a vector via

$$\alpha_1.w_1 + \alpha_2.w_2 \cdots + \alpha_m.w_m := \big(w_1, v_2, \ldots, w_m\big) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}.$$

We can do the same for each of the $L(v_i)$:

$$L(v_1) = \big(w_1, w_2, \ldots, w_m\big) \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, L(v_2) = \big(w_1, w_2, \ldots, w_m\big) \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, L(v_n) = \big(w_1, w_2, \ldots, w_m\big) \begin{pmatrix} a_{n1} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Its better to arrange this information with $L(v_i)$ as a list so that the column vectors appear as columns of an $m \times n$ array, *i.e.*

$$\big(L(v_1), L(v_2), \ldots, L(v_n)\big) = \big(w_1, w_2, \ldots, w_m\big) \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

The array on the right is exactly the matrix of L in the bases given. As an example suppose V was dimension 3 and W was dimension 2 and you knew

$$L(v_1) = 2w_1 + 3w_2\,, \quad L(v_2) = 5w_2\,, \quad L(v_3) = -w_1 + w_2$$

then we would write

$$\begin{aligned}\big(L(v_1), L(v_2), L(v_3)\big) &= (2w_1 + 3w_2, 5w_2, -w_1 + w_2) \\ &= \left((w_1, w_2)\begin{pmatrix}2\\3\end{pmatrix}, (w_1, w_2)\begin{pmatrix}0\\5\end{pmatrix}, (w_1, w_2)\begin{pmatrix}-1\\1\end{pmatrix}\right) \\ &= (w_1 w_2)\begin{pmatrix}2 & 0 & -1\\3 & 5 & 1\end{pmatrix}\end{aligned}$$

So the matrix of L IN THESE BASES is $\begin{pmatrix}2 & 0 & -1\\3 & 5 & 1\end{pmatrix}$.

Compute the matrix of L as above but in new bases $(v_1 + v_2, v_2 + v_3, v_3 + v_1)$ and $(w_1 + w_2, w_2 - w_1)$.

Here's another example to try. Let $L : \mathbb{C}_3[z] \to \mathbb{C}^3$ via

$$L(a_0 + a_1 z + a_2 z^2 + a_3 z^3) = (a_0 + ia_3, a_1 - ia_2, a_1 + a_2 + a_3)\,.$$

Compute the matrix of L in bases $(1, z, z^2, z^3)$ and $(1, 1, 0), (0, 1, 1), (1, 0, 1)$.

If $L : V \to W$ has a 5×3 matrix (5 rows, 3 columns) in some choice of bases for V and W, what are the dimensions of W and V?

Finally, suppose $L : V \to W$ and $T : W \to U$ and bases are $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_m)$ and $(u_1, \ldots, u_l)$. Moreover let

$$L(v_i) = \sum_{j=1}^{m} a_{ji} w_j \quad \& \quad T(w_i) = \sum_{j=1}^{l} b_{ji} u_j\,.$$

Compute the matrices of L, T and $T \circ L$. Does the matrix of $L \circ T$ make sense?