3 Let $n \in \mathbb{Z}_+$ be a positive integer and $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ be defined by

$$T(x_1, ..., x_n) = (x_1 + ... + x_n, ..., x_1 + ... + x_n).$$

Compute the eigenvalues and associated eigenvectors for T.

Solution The range of T is the space of vectors with all components equal, thus the eigenvectors of T are constant vectors. I find the eigenvalue by considering the action of T on $(1, ..., 1)$, which is $(1 + ... + 1, ..., 1 + ... + 1) = (n, ..., n)$. Thus n is an eigenvalue of T with multiplicity n and that $(1, ..., 1)$ is an eigevector.

4 Find the eigenvalues and associated eigenvectors for the linear operators on $\mathbb{F}^2$ defined by each given 2×2 matrix

$$(a) \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \quad (b) \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \quad (c) \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$$

$$(d) \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix} \quad (e) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (f) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solution (a) The characteristic polynomial is $(3 - \lambda)(-1 - \lambda) = 0$, which has solutions $\lambda = 3, -1$. To find the eigenvectors I must solve the linear equation $(M - \lambda I)x = 0$. For $\lambda = 3$ this corresponds to

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $8x - 4y = 0$ or $2x = y$, so $(1, 2)$ is an eigenvector with eigenvalue 3. For $\lambda = -1$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $4x = 0$, or $x = 0$ so $(0, 1)$ is an eigenvector with eigenvalue -1.

(b)The characteristic polynomial is $(10 - \lambda)(-2 - \lambda) + 36 = 0$, which has solution $\lambda = 4$. To find the eigenvectors I must solve the linear equation $(M - \lambda I)x = 0$. For $\lambda = 4$ this corresponds to

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $6x - 9y = 0$, so $(9, 6)$ is an eigenvector with eigenvalue 4.

(c) The characteristic polynomial is $\lambda^2 - 12 = 0$, which has solutions $\pm 2\sqrt{3}$. To find the eigenvectors I must solve the linear equation $(M - \lambda I)x = 0$. For $\lambda = 2\sqrt{3}$ this corresponds to

$$\begin{pmatrix} -2\sqrt{3} & 3 \\ 4 & -2\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $-2\sqrt{3}x + 3y = 0$ so $(3, 2\sqrt{3})$ is an eigenvector with eigenvalue $2\sqrt{3}$. For $\lambda = -2\sqrt{3}$ this corresponds to

$$\begin{pmatrix} 2\sqrt{3} & 3 \\ 4 & 2\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $2\sqrt{3}x + 3y = 0$, so $(3, -2\sqrt{3})$ is an eigenvector with eigenvalue $-2\sqrt{3}$.

(d) The characteristic polynomial is $(-2-\lambda)(2-\lambda)+7=0$, which has solutions $\lambda = \pm\sqrt{3}i$. To find the eigenvectors I must solve the linear equation $(M-\lambda I)x = 0$. For $\lambda = \sqrt{3}i$ this corresponds to

$$\begin{pmatrix} -2-\sqrt{3}i & -7 \\ 1 & 2-\sqrt{3}i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $-(2+\sqrt{3}i)x - 7y = 0$ so $(-7, 2+\sqrt{3}i)$ is an eigenvector with eigenvalue $\sqrt{3}i$. For $\lambda = -\sqrt{3}$ this corresponds to

$$\begin{pmatrix} -2+\sqrt{3}i & -7 \\ 1 & 2+\sqrt{3}i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $(-2+\sqrt{3}i)x - 7y = 0$, so $(7, 2-\sqrt{3}i)$ is an eigenvector with eigenvalue $-\sqrt{3}i$.

(e) The characteristic polynomial is $\lambda^2 = 0$, which has solution $\lambda = 0$. The all zero matrix is the matrix representation of the zero operators, so every vector is an eigenvector with eigenvalue zero.

(f) The characteristic polynomial is $(\lambda-1)^2 = 0$, which has solution $\lambda = 1$. This is the matrix representation of the identity operator, so every vector is an eigenvector with eigenvalue 1.

5 For each Matrix A below, find eigenvalues for the induced linear operator T on $\mathbb{F}^n$ without performing any calculations. Then describe the eigenvectors $v \in \mathbb{F}^n$ associated to each eigenvalue λ but looking at solutions to the matrix equation $(A-\lambda I)v = 0$, where I denotes the identity map on $\mathbb{F}^n$.

$$(a)\begin{pmatrix} -1 & 6 \\ 0 & 5 \end{pmatrix} \quad \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution The key to this problem is that the diagonal entries of a triangular matrix are the eigenvalues of that matrix.

(a) The eigenvalues are $-1, 5$. The matrix equations are

$$\begin{pmatrix} 0 & 6 \\ 0 & 6\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -6 & 6 \\ 0 & 0\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which have solutions $y = 0$ and $-x + y = 0$ respectively, so $(1, 0)$ is an eigenvector with eigenvalue -1 and $(1, -1)$ is an eigenvector with eigenvalue 5.

(b) The eigenvalues are $-\frac{1}{3}, 1$ and $\frac{1}{2}$. The matrix equations are

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -\frac{4}{3} & 0 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -\frac{5}{3} & 0 & 0 & 0 \\ 0 & -\frac{5}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which have solutions $x_1 = x_2 = 0$, $x_3 = 0$, and $x_4 = 0$ respectively. Thus $(0,0,1,0)$ and $(0,0,0,1)$ span the space of eigenvectors with eigenvalue $-\frac{1}{3}$, $(0,0,1,0)$ is an eigenvector with eigenvalue 1, and $(0,0,0,1)$ is an eigenvector with eigenvalue $\frac{1}{2}$.

(c) The eigenvalues are 1, $\frac{1}{2}$, 0, and 2. The matrix equations are

$$\begin{pmatrix} 0 & 3 & 7 & 11 \\ 0 & -\frac{1}{2} & 3 & 8 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{1}{2} & 3 & 7 & 11 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -1 & 3 & 7 & 11 \\ 0 & -\frac{3}{2} & 3 & 8 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which have solutions $x_4 = 0x_3 = 0x_2 = 0$, $x_4 = x_3 = 0$ and $\frac{1}{2}x_1 + 3x_2 = 0$, $x_4 = 0$and $\frac{1}{2}x_2 + 3x_3 = 0$ and $x_1 - 6x_3 + 7x_3 = 0$, and $-2x_3 + 4x_4 = 0$ and $-\frac{3}{2}x_2 + 3x_3 + 8x_4 = 0$, $-x_1 + 3x_3 + 7x_2 + 11x_4 = 0$. Thus $(1,0,0,0)$ is an eigenvector with eigenvalue 1, $(3, -\frac{1}{2}, 0, 0)$ is an eigenvector with eigenvalue $\frac{1}{2}$, $(-1,-6,1,0)$ is an eigenvector with eigenvalue zero, and $(43, \frac{28}{3}, 2, 1)$ is an eigenvector with eigenvalue 2.

6 For each matrix A below, describe the invariant subspace for the induced linear operator T on $\mathbb{F}^2$ that maps each $v \in \mathbb{F}^2$ to $T(v) = Av$

$$(a)\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}, \quad (b)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (c)\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \quad (d)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Solution (a) The characteristic polynomial is $(4-\lambda)(1-\lambda)+2 = 0$ so the eigenvalues are $3, 2$. To find the eigenvalues I must solve the matrix equations

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $(1,1)$ is an eigenvector with eigenvalue 3 and $(\frac{1}{2},1)$ is an eigenvector with eigenvalue 2.

(b) The characteristic polynomial is $\lambda^2+1=0$ so $i,-i$ are the eigenvalues. To find the eigenvectors I must solve the matrix equation

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

and

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

7 Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix}, \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

Define two real numbers λ_+ and λ_- as follows:

$$\lambda_+ = \frac{1+\sqrt{5}}{2}, \qquad \lambda_- = \frac{1-\sqrt{5}}{2}.$$

(a) Find the matrix of T with respect to the canonical basis of $\mathbb{R}^2$.

(b) Verify that λ_+ and λ_- are eigenvalues of T by showing that v_+ and v_- are eigenvectors, where

$$v_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}, \qquad v_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}.$$

(c) Show that (v_+, v_-) is a basis of $\mathbb{R}^2$.

(d) Find the matrix of T with respect to the basis (v_+, v_-) for $\mathbb{R}^2$

Solution (a) The matrix of a linear transformation has as its columns the image of the basis vectors, hence

$$M[T] = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(b) I perform the matrix multiplication

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} \end{pmatrix}$$

which is $\lambda_+ v_+$. Next I perform the matrix multiplication

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} \end{pmatrix}$$

which is $\lambda_- v_-$.

(c) Since $\mathbb{R}^2$ is two dimensional it suffices to show that (v_+, v_-) is an independent set. I check linear independence using row reduction

$$\begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

which reduces to

$$\begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & -\sqrt{5} \end{pmatrix}$$

which is an invertible matrix; hence, (v_+, v_-) are linearly independent.

(d) Conjugation by a change of basis transformation allows me to go from the answer to (a) to the to (d). The matrix to change from the eigen-basis to the standard basis is

$$S[T] = \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix},$$

which has inverse

$$\frac{\sqrt{5}}{5}\begin{pmatrix} \frac{1-\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix};$$

thus

$$T' = STS^{-1}$$

which is

$$\frac{\sqrt{5}}{5}\begin{pmatrix} -3-\sqrt{5} & \frac{1+\sqrt{5}}{2} \\ -\frac{1}{2}-\sqrt{5} & \frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

1 Let V be a finite-dimensional vector space over $\mathbb{F}$ with $T \in \mathcal{L}(V,V)$, and let $U_1, ..., U_m$ be subspaces of V that are invariant under T. Prove that $U_1 + ...U_m$ must then also be an invariant subspace of V under T.

Solution Let $v \in U_1 + ... + U_m$, then there exists $w_i \in U_i$ and $\alpha_i \in \mathbb{F}$ such that

$$v = \sum_{i=1}^{m} \alpha_i w_i.$$

Applying T to v yields

$$T(v) = \sum_{i=1}^{m} \alpha_i T(w_i).$$

Since the U_i are fixed by T the $T(w_i) \in U_i$ so $T(v) \in U_1 + ...U_m$

2 Let V be a finite-dimensional vector space over $\mathbb{F}$ with $T \in \mathcal{L}(V,V)$, and suppose that U_1 and U_2 are subspaces of V that are invariant under T. Prove that $U_1 \cap U_2$ is also an invariant subspace of V under T.

Solution Let $v \in U_1$ and $v \in U_2$, then $v \in U_1 \cap U_2$. Since U_1 is an invariant subspace of T then $Tv \in U_1$; likewise, $Tv \in U_2$. Thus T takes elements of $U_1 \cap U_2$ to elements of $U_1 \cap U_2$, so $U_1 \cap U_2$ is an invariant subspace.

3 Let V be a finite-dimensional vector space over $\mathbb{F}$ with $T \in \mathcal{L}(V,V)$ invertible and $\lambda \in \mathbb{F}\backslash\{0\}$. Prove λ is an eigenvalue for T if and only if λ^{-1} is an eigenvalue for T^{-1}.

Solution First I show that λ^{-1} is an eigenvalue of T^{-1} if T is an eigenvalue of λ. Let v be an eigenvalue with eigenvalue λ then

$$\begin{aligned} v =& T^{-1}Tv \\ =& \lambda T^{-1}v. \end{aligned}$$

Now I show that if λ^{-1} is an eigenvalue of T^{-1} then λ is an eigenvalue of T. Let v be an eigenvalue with eigenvalue λ then

$$\begin{aligned} v =& TT^{-1}v \\ =& \lambda^{-1}Tv. \end{aligned}$$

4 Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V,V)$ has the property that every $v \in V$ is an eigenvector for T. Prove that T must then be a scalar multiple of the identity function on V.

Solution First I show that T has only one eigenvalue. Let u, v be linearly independent vectors, then

$$\begin{aligned} T(u+v) =& \lambda_1 u + \lambda_2 v \\ =& \lambda_3(u+v). \end{aligned}$$

Thus

$$(\lambda_1 - \lambda_3)u + (\lambda_2 - \lambda_3)v = 0,$$

since u, v are linearly independent it follows that $\lambda_1 = \lambda_3 = \lambda_2$. Since $T(v) = \lambda v$ for all v the operator T acts as λ times the identity operator.

5 Let V be a finite-dimensional vector space over $\mathbb{F}$, and let $S, T \in \mathcal{L}(V)$ be linear operators on V with S invertible. Given any polynomial $p(z) \in \mathbb{F}[z]$, prove that

$$p(S \circ T \circ S^{-1}) = S \circ p(T) \circ S^{-1}.$$

Solution First I prove by induction that on n that the claim holds for polynomials of the form z^n. In the base case of a constant polynomial

$$\begin{aligned} P(S \circ T \circ S^{-1}) =& a_0 \\ =& a_0 SS^{-1} \\ =& S \circ a_0 I \circ S^{-1} \\ =& S \circ P(T) \circ S^{-1}. \end{aligned}$$

Now suppose that for $n < N$ we have that

$$(S \circ T \circ S^{-1})^n = S \circ T^n \circ S^{-1}$$

and consider

$$\begin{aligned}(S \circ T \circ S^{-1})^N =&(S \circ T \circ S^{-1})^{N-1} \circ (S \circ T \circ S^{-1})\\ =&S \circ T^{N-1} \circ S^{-1} \circ S \circ T \circ S^{-1}\\ =&S \circ T^{N-1} \circ T \circ S^{-1}\\ =&S \circ T^N \circ S^{-1},\end{aligned}$$

which completes the proof. Now let $p(z) \in \mathbb{F}[z]$ be given, then $P(z) = \sum_{i=0}^n a_i z^i$ and

$$\begin{aligned}P(S \circ T \circ S^{-1}) =& \sum_{i=0}^n a_n (S \circ T \circ S^{-1})^n\\ =& \sum_{i=0}^n a_n S \circ T^n \circ S^{-1}\end{aligned}$$
$$= S \circ P(T) \circ S^{-1}.$$

8 Prove or give a counterexample to the following claim:

Claim. *Let V be a finite-dimensional vector space over $\mathbb{F}$, and let $T \in \mathcal{L}(V)$ be a linear operator on V. If the matrix for T with respect to some basis on V has all zeros on the diagonal, then T is not invertible.*

Solution This is false, the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has all zeroes on the diagonal, but is its own inverse.

12

(a) Let $a, b, c, d \in \mathbb{F}$ and consider the system of equations given by

$$ax_1 + bx + 2 = 0$$

$$cx_1 + dx_2 = 0.$$

Note that $x_1 = x_2 = 0$ is a solution for any choice of a, b, c, and d. Prove that this system of equations has a non-trivial solution if and only if $ad - bc = 0$.

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2x2}$, and recall that we can define a linear operator $T \in \mathcal{L}(\mathbb{F}^2)$ on $\mathbb{F}^2$ by setting $T(v) = Av$ for each $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{F}^2$. Show that the eigenvalues for T are exactly the $\lambda \in \mathbb{F}$ for which $p(\lambda) = 0$, where $p(z) = (a - z)(d - z) - bc$.

Solution (a) Substituting $ax_1 + bx_2 = 0$ into $cx_1 + dx_2 = 0$ yields $a(-\frac{d}{c}x_2) + bx_2 = 0$ or $(ad - bc)x_2 = 0$, so when $ad - bc \neq 0$ we have $x_2 = 0$, which implies $x_1 = 0$. On the other hand, when $ad - bc = 0$ we may choose x_2 arbitrarily and solve for x_1 from the original system.

(b) An eigenvalue of A is a non-zero solution to the equation $(A - \lambda I) = 0$, which has matrix form

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From part (a) there is a non-zero solution if and only if $(a - \lambda)(d - \lambda) - bc = 0$, thus the eigenvalues of A are the zeroes of $P(z) = a - z)(d - z) - bc = 0$.