Chapter 7 WORKSHEET

① Let $L : V \xrightarrow[\text{linear}]{} V$, V a vector space over $\mathbb{C}$, $\dim V = n$.

(i) Suppose $0 \neq v \in V$. Explain why there must exist a relationship

$$\alpha_0 v + \alpha_1 L v + \alpha_2 L^2 v + \cdots + \alpha_n L^n v = 0$$

with not all α's vanishing.

(ii) True or false: For some $m > 0$

$$\alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m = \alpha_m (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)$$

with $\alpha_m \neq 0$ & $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$.

Why?

(iii) True or false:

For some $i \in \{1, 2, \dots, m\}$, $L - \lambda_i Id$ is <u>NOT</u> 1:1.

Why?

(iv) Show that every $L : V \xrightarrow[\text{near}]{} V$, $\{0\} \neq V$ a vector space over $\mathbb{C}$, has at least one eigenvalue.

2 Let $L : V \xrightarrow[\text{linear}]{} V$, $\{0\} \neq V$ finite dimensional vector space over $\mathbb{C}$.

(i) Let λ be an eigenvalue with associated eigenvector v.

Show that $M[L]$ in a basis $\{v, f_2, f_3, \dots, f_{\dim V}\}$ has the form

$$M[L] = \left(\begin{array}{c|c} \lambda & N \\ \hline 0 & \\ 0 & \hat{M} \\ \vdots & \\ 0 & \end{array}\right)$$

where N is a $1 \times (\dim V - 1)$ matrix and $\hat{M}$ is a $(\dim V - 1) \times (\dim V - 1)$ matrix.

(ii) Let $\hat{V} = \operatorname{span}(f_2, \dots, f_{\dim V})$ and $\hat{L} : \hat{V} \to \hat{V}$ defined by $\hat{L} f_j = \sum_{k=2}^{\dim V} \hat{M}_j^k f_k$.

Explain why $\hat{V}$ has a basis in which $M[\hat{L}]$ takes the form

$$M[\hat{L}] = \left(\begin{array}{c|c} \hat{\lambda} & \\ 0 & * \\ \vdots & \\ 0 & \end{array}\right)$$

where $\hat{\lambda}$ is an eigenvalue of $\hat{L}$.

(iii) Use (ii) and induction to show there exists a basis in which $M[L]$ is <u>upper triangular</u>, i.e. $M[L] = \begin{pmatrix} * & & * \\ & * & \\ 0 & & \ddots \\ & & & * \end{pmatrix}$

(iv) Suppose $\lambda \neq 0$ in the above and $\hat{v}$ is the eigenvector of $\hat{L}$ associated to $\hat{\lambda}$. Explain why

$$L(\hat{v}) = \alpha v + \hat{\lambda}\hat{v} \quad \text{for some } \alpha.$$

(v) Compute $L(\hat{v} - \frac{\alpha}{\lambda} v)$.

(vi) Is $\hat{\lambda}$ an eigenvalue of L? Why

(vii) What can you say about the diagonal elements of the upper triangular matrix $M[L]$ of part (iii) when L is invertible?