Chapter 1.

CE1.

(a)
$$\begin{aligned} x + 2y - 2z + 3w &= 2 \quad (-2) \quad (-5) \\ 2x + 4y - 3z + 4w &= 5 \\ 5x + 10y - 8z + 11w &= 12 \end{aligned}$$

$$\Rightarrow \begin{aligned} x + 2y - 2z + 3w &= 2 \\ z - 2w &= 1 \quad (-2) \\ 2z - 4w &= 2 \end{aligned} \qquad \Rightarrow \begin{aligned} x + 2y - 2z + 3w &= 2 \\ z - 2w &= 1 \\ 0 &= 0 \end{aligned}$$

Let. $y = r$, $w = s$

Solution
$$\begin{cases} x = -2r + s + 4 \\ y = r \\ z = 2s + 1 \\ w = s \end{cases}$$

$$f: R^4 \to R^3 \qquad f(x, y, z, w) = \begin{pmatrix} x + 2y - 2z + 3w \\ 2x + 4y - 3z + 4w \\ 5x + 10y - 8z + 11w \end{pmatrix}$$

b).
$$\begin{aligned} x + 2y - 3z &= 4 \quad (-1) \quad (-2) \\ x + 3y + z &= 11 \\ 2x + 5y - 4z &= 13 \\ 2x + 6y + 2z &= 22 \end{aligned} \qquad \Rightarrow \begin{aligned} x + 2y - 3z &= 4 \\ y + 4z &= 7 \quad (-2) \\ y + 2z &= 5 \\ 2y + 8z &= 14 \end{aligned}$$

$$\Rightarrow \begin{aligned} x + 2y - 3z &= 4 \\ y + 4z &= 7 \\ -2z &= -2 \\ 0 &= 0 \end{aligned} \qquad \Rightarrow \begin{cases} x = 1 \\ y = 3 \\ z = 1 \end{cases}$$

$$f: R^3 \to R^4 \qquad f(x, y, z) = \begin{pmatrix} x + 2y - 3z \\ x + 3y + z \\ 2x + 5y - 4z \\ 2x + 6y + 2z \end{pmatrix}$$

(c). $x+2y-3z=-1$ (-3) (-5)
$3x-y+2z=7$
$5x+3y-4z=2.$

$$\Rightarrow \begin{cases} x+2y-3z=-1 \\ -7y+11z=10. \quad (-1) \\ -7y+11z=7 \end{cases}$$

$$\Rightarrow \begin{cases} x+2y-3z=-1 \\ -7y+11z=10. \\ 0=-3 \quad \text{Contradiction.} \end{cases}$$

Solution doesn't exist.

$$f: R^3 \to R^3 \qquad f(x,y,z) = \begin{pmatrix} x+2y-3z \\ 3x-y+2z \\ 5x+3y-4z \end{pmatrix}$$

3. $x_1+3x_2=2.$ (-1)
$x_1-x_2=1.$

$$\Rightarrow \begin{cases} x_1+3x_2=2. \\ -4x_2=-1 \end{cases} \Rightarrow \begin{cases} x_1=\frac{5}{4} \\ x_2=\frac{1}{4} \end{cases}$$

PWE 1.

$$\begin{cases} ax_1+bx_2=0. \\ cx_1+dx_2=0 \end{cases}$$

Prove that if $ad-bc\neq 0$, then $x_1=x_2=0$ is the only solution.

Proof. Given $ad-bc\neq 0$.

① if $a=0$, $bc\neq 0$, both $b\neq 0$ and $c\neq 0$.

$$\begin{cases} bx_2=0. \\ cx_1+dx_2=0. \end{cases} \quad \begin{matrix}\text{then} \\ b\neq 0\end{matrix} \begin{cases} x_2=0. \\ cx_1=0 \end{cases} \overset{c\neq 0}{\Rightarrow} \begin{cases} x_2=0 \\ x_1=0. \end{cases}$$

② if $a\neq 0$.

$$\begin{cases} ax_1+bx_2=0 \quad \left(-\frac{c}{a}\right) \\ cx_1+dx_2=0 \end{cases} \Rightarrow \begin{cases} ax_1+bx_2=0. \\ \left(d-\frac{bc}{a}\right)x_2=0. \end{cases}$$

since $ad-bc \neq 0$. $\Rightarrow d-\frac{bc}{a} \neq 0$., then $x_2=0$. and $x_1=0$.

Chapter 4 CE 1.

(a) Yes.

① Commutativity: $u, v \in R$ $u+v=v+u$.

② Associativity: $u, v, w \in R$ $(u+v)+w=u+(v+w)$.

$(ab)v=(a)(bv)$.

③ Additive identity: $0$.

④ Additive inverse: additive inverse of $v$ is $-v$.

⑤ Multiplicative Identity: $1$.

⑥ Distributivity: $a(u+v) = au+av$.

$(a+b)u = au+bu$.

(b) $\{(x, 0) \in R\}$.

Yes.

①. $(x,0)+(y,0) = (x+y,0) = (y,0)+(x,0)$.

②. $\left((x,0)+(y,0)\right)+(z,0) = (x+y+z,0) = (x,0)+\left((y,0)+(z,0)\right)$.

$(ab)(x,0) = (abx,0) = a\left(b(x,0)\right)$.

③ $(0,0)$

④. additive inverse of $(x,0)$ is $(-x,0)$.

⑤. $1$.

⑥. $a\left((x,0)+(y,0)\right) = (ax+ay,0) = a(x,0)+a(y,0)$.

$(a+b)(x,0) = ((a+b)x,0) = (ax+bx,0) = a(x,0)+b(x,0)$.

(c) $\{(x,1)\}$.

No, because there is no additive identity.

$(0,0) \notin \{(x,1) \mid x \in R\}$.

(d) $\{(x,0) \mid x \geq 0\}$.

No, because the additive inverse of $(1,0)$ doesn't exist

$(-1,0) \notin \{(x,0) \mid x \geq 0\}$.

(e) $\{(x,1) \mid x \geq 0\}$.

No, there is no additive identity.

(f). $\left\{ \begin{pmatrix} a & a+b \\ a+b & a \end{pmatrix} \middle| \ a,b \in R \right\}$.

①. $\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & a_1+a_2+b_1+b_2 \\ a_1+a_2+b_1+b_2 & a_1+a_2 \end{pmatrix}$.

$= \begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & a_2 \end{pmatrix}$

②. $\left(\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & a_2 \end{pmatrix}\right) + \begin{pmatrix} a_3 & a_3+b_3 \\ a_3+b_3 & a_3 \end{pmatrix} = \begin{pmatrix} a_1+a_2+a_3 & a_1+a_2+a_3+b_1+b_2+b_3 \\ a_1+a_2+a_3+b_1+b_2+b_3 & a_1+a_2+a_3 \end{pmatrix}$

$= \begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & a_1 \end{pmatrix} + \left(\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & a_2 \end{pmatrix} + \begin{pmatrix} a_3 & a_3+b_3 \\ a_3+b_3 & a_3 \end{pmatrix}\right)$

③ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

④ additive inverse of $\begin{pmatrix} a & a+b \\ a+b & a \end{pmatrix}$ is $\begin{pmatrix} -a & -a-b \\ -a-b & -a \end{pmatrix}$.

⑤ 1

⑥ $k\left(\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & a_2 \end{pmatrix}\right) = \begin{pmatrix} ka_1+ka_2 & ka_1+ka_2+kb_1+kb_2 \\ ka_1+ka_2+kb_1+kb_2 & ka_1+ka_2 \end{pmatrix}$.

$= k\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & a_1 \end{pmatrix} + k\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & a_2 \end{pmatrix}$

$(k_1+k_2)\begin{pmatrix} a & a+b \\ a+b & a \end{pmatrix} = \begin{pmatrix} k_1a+k_2a & k_1a+k_2a+k_1b+k_2b \\ k_1a+k_2a+k_1b+k_2b & k_1a+k_2a \end{pmatrix}$

$= k_1\begin{pmatrix} a & a+b \\ a+b & a \end{pmatrix} + k_2\begin{pmatrix} a & a+b \\ a+b & a \end{pmatrix}$.

(9). $\left\{\begin{bmatrix} a & a+b+1 \\ a+b & a \end{bmatrix}\right\}$.

No, because there is no additive identity.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin \left\{\begin{bmatrix} a & a+b+1 \\ a+b & a \end{bmatrix}\right\}$.

2. $V = \{(x_1, x_2, x_3) \mid x_1+2x_2+2x_3 = 0\}$.

① $(0,0;0) \in V$.

②. Take $(x_1, x_2, x_3)$ $(y_1, y_2, y_3)$

such that $x_1+2x_2+2x_3=0$ $y_1+2y_2+2y_3=0$.

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$

$x_1+y_1+2(x_2+y_2)+2(x_3+y_3) = x_1+2x_2+2x_3+y_1+2y_2+2y_3$

$=0+0=0$.

so, $(x_1, x_2, x_3)+(y_1, y_2, y_3) \in V$.

③. $k(x_1, x_2, x_3) = (kx_1, kx_2, kx_3)$

$kx_1 + 2kx_2 + 2kx_3 = k(x_1 + 2x_2 + 2x_3) = 0$.

so $k(x_1, x_2, x_3) \in V$.

PWE 1.

Given $av = 0$. If $a \neq 0$. $a^{-1}av = a^{-1}0$.

$\Rightarrow v = 0$.

Therefore, either $a = 0$ or $v = 0$.