Chapter 5

CE 3.

(a) $d=3$ basis: $(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$

(b) $d=3$ basis $(1,0,0,1)$ $(0,1,0,1)$ $(0,0,1,0)$

(c). $d=2$ basis. $(1,0,1,1)$ $(0,1,-1,1)$

(d) $d=2$. $\begin{cases} x_4 = x_1+x_2 & ① \\ x_3 = x_1 - x_2 & ② \\ x_3+x_4 = 2x_1 & ③ \end{cases}$ Adding ① and ② yields ③, so ③ is redudant.

Basis $(1,0,1,1)$ $(0,1,-1,1)$

(e) $d=1$ basis $(1,1,1,1)$

CE 4.

(a) $(\lambda,-1,-1)$, $(-1,\lambda,-1)$, $(-1,-1,\lambda)$ is linearly dependent if there exists c_1, c_2, c_3 (not all 0) such that.

$$c_1(\lambda,-1,-1) + c_2(-1,\lambda,-1) + c_3(-1,-1,\lambda) = 0$$

$$(c_1\lambda - c_2 - c_3,\ -c_1+\lambda c_2 - c_3,\ -c_1-c_2+\lambda c_3) = 0$$

$$\begin{cases} c_1\lambda - c_2 - c_3 = 0 \\ -c_1 + \lambda c_2 - c_3 = 0 & \textcircled{\lambda}\ \textcircled{-1} \\ -c_1 - c_2 + \lambda c_3 = 0. \end{cases} \quad \begin{cases} (\lambda^2-1)c_2 - (\lambda+1)c_3 = 0 \\ -c_1 + \lambda c_2 - c_3 = 0 & \textcircled{-1} \\ -(\lambda+1)c_2 + (\lambda+1)c_3 = 0. & \textcircled{-1} \end{cases}$$

$$\Rightarrow \begin{cases} c_1 - \lambda c_2 + c_3 = 0 \\ (\lambda+1)c_2 - (\lambda+1)c_3 = 0 & \textcircled{\lambda-1} \\ (\lambda+1)(\lambda-1)c_2 - (\lambda+1)c_3 = 0 \end{cases} \quad \begin{cases} c_1 - \lambda c_2 + c_3 = 0 \\ (\lambda+1)c_2 - (\lambda+1)c_3 = 0 \\ [(\lambda+1)(\lambda-1) - (\lambda+1)]c_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 - \lambda c_2 + c_3 = 0 \\ (\lambda+1)c_2 - (\lambda+1)c_3 = 0 \\ (\lambda+1)(\lambda-2)c_3 = 0 \end{cases}$$

This linear system has a nonzero solution if $(\lambda+1)(\lambda-2) = 0$.

Therefore, $\lambda = -1$ or 2.

(b). $\sin^2(x)$, $\cos(2x)$, λ.

Let $c_1\sin^2 x + c_2\cos 2x + c_3\lambda = 0$.

Remember. $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

$$c_1 \frac{1-\cos 2x}{2} + c_2 \cos 2x + c_3 \lambda = 0.$$

$$\left(\frac{c_1}{2} + c_3\lambda\right) + \left(c_2 - \frac{c_1}{2}\right) \cos 2x = 0.$$

Then: $\frac{c_1}{2} + \lambda c_3 = 0.$

$c_2 - \frac{c_1}{2} = 0.$

There are 3 variables: c_1, c_2, c_3 and 2 equations. No matter what λ is, there are infinitely many nonzero solutions. $\{\sin^2 x, \cos 2x, \lambda\}$ is linearly dependent for any $\lambda \in R$.

CE5.

(a) $\dim V = 4$ basis $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

(b). Three vectors can't span a 4 dimensional space.

$$\text{span}((1,1,0,0), (0,1,1,0), (0,0,1,1)) \neq V.$$

For ~~exp~~ example, $\begin{pmatrix}1\\1\\1\\0\end{pmatrix} \notin \text{span}((1,1,0,0), (0,1,1,0), (0,0,1,1))$.

Let $\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = c_1 \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + c_3 \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$

$c_1 = 1$ $c_2 = 0$ $c_3 =$ both 1 & 0 contradictory!

Such c_1, c_2, c_3 doesn't exist.

(c). $c_1(1,-1,0,0) + c_2(0,1,-1,0) + c_3(0,0,1,-1) + c_4(-1,0,0,1) = (0,0,0,0)$

$$\begin{cases} c_1 - c_4 = 0 \\ -c_1 + c_2 = 0 \\ -c_2 + c_3 = 0 \\ -c_3 + c_4 = 0 \end{cases} \qquad \begin{cases} c_1 - c_4 = 0 \\ c_2 - c_4 = 0 \\ -c_2 + c_3 = 0 \\ -c_3 + c_4 = 0 \end{cases} \qquad \begin{cases} c_1 - c_4 = 0 \\ c_2 - c_4 = 0 \\ c_3 - c_4 = 0 \\ -c_3 + c_4 = 0 \end{cases}$$

$$\begin{cases} c_1 \quad\quad - c_4 = 0 \\ \quad c_2 \quad - c_4 = 0 \\ \quad\quad c_3 - c_4 = 0. \\ \quad\quad\quad 0 = 0. \end{cases}$$

Nonzero solution exists, These vectors are linearly dependent.

(d). Prove by contradiction.

If $\{v_1, v_2, v_3, v_4\}$ is linearly dependent, one vector must be the linear combination of other vectors. For instance, $v_4 = c_1 v_1 + c_2 v_2 + c_3 v_3$.

Then $\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3\}$

Three vectors can't span R^4, so $\text{span}\{v_1, v_2, v_3\} \neq V$.

Contradiction!

(e). Let $U = \text{span}\{v_1, v_2\}$, which is 2 dimensional.

There exists a subspace W such that $\longleftarrow$ Theorem 5.4.5.

$$R^4 = U \oplus W.$$

$\dim(R^4) = \dim(U) + \dim(W)$. so $\dim(W) = 2$.

Let $\{u, w\}$ be a basis of W, then $\{v_1, v_2, u, w\}$ is a basis of R^4.

PWE. 3. There exists $c_1, c_2 \cdots c_n$ (not all 0) such that.

$$c_1(v_1+w) + c_2(v_2+w) + \cdots + c_n(v_n+w) = 0.$$

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n + (c_1+c_2+\cdots+c_n)w = 0.$$

Claim: $c_1+c_2+\cdots+c_n \neq 0$, other-wise,

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

Since $\{v_1 \cdots v_n\}$ is linearly independent, $c_1, c_2 \cdots c_n = 0$.

$(c_1+c_2+\cdots+c_n)\neq 0.$

$$W = -\frac{c_1V_1 + c_2V_2 + \cdots + c_nV_n}{c_1+c_2+\cdots+c_n}$$

$\Rightarrow W \in \text{span}\{V_1, V_2 \cdots V_n\}.$

PWE 4. Let $\{u_1, u_2 \cdots u_n\}$ be a basis of V.

$U_1 = \text{span}\{u_1\}$, $U_2 = \text{span}\{u_2\}$, $\cdots$ $U_n = \text{span}\{u_n\}$.

First. $V = U_1 + U_2 + \cdots + U_n$.

for every $v \in V$, there exists $\{c_1 \cdots c_n\}$ s.t.

$v = c_1u_1 + c_2u_2 + \cdots + c_nu_n.$

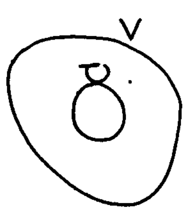
Second: $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$, since $U_i \cap U_j = \{0\}$ for all i, j.

Let $w = U_i \cap U_j$.

$$\left.\begin{array}{l} \text{then } w = c_iu_i \\ \quad\;\; w = c_ju_j \end{array}\right\} \Rightarrow c_iu_i - c_ju_j = 0.$$

$\{u_i, u_j\}$ is linearly independent, so $c_i = 0$ $c_j = 0$.

$w = 0$.

PWE 5. 

Let $\dim(U) = \dim(V) = n$.

Let $\{u_1 \cdots u_n\}$ be a basis of U.

Let $\{v_1 \cdots v_n\}$ be a basis of V.

Since $U \subseteq V$, According to the Basis Extension Theorem,

$\{u_1 \cdots u_n\}$ can be extended to a basis of V.

However, $n = \dim(V)$, $\{u_1 \cdots u_n\}$ is already a basis of V.

$U = V$.

PWE. 6. $\dim(F_m[z]) = m+1$, since $\{1, z, z^2 \cdots z^m\}$ is a basis.

Let $U = \{ p(z) \in F_m[z] \mid p(2) = 0 \}$.

U is a subspace of $F_m[z]$.

What's the dimension of U?

Let $p(z) \in U$ and $p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$.

$p(2) = 0$, then $a_m 2^m + a_{m-1} 2^{m-1} + \cdots + 2a_1 + a_0 = 0$

$$\Rightarrow a_0 = -a_m 2^m - a_{m-1} 2^{m-1} - \cdots - 2a_1$$

$$p(z) = a_m (z^m - 2^m) + a_{m-1}(z^{m-1} - 2^{m-1}) + \cdots + a_1 (z-2)$$

$\{(z^m - 2^m), z^{m-1} - 2^{m-1}, \ldots, z-2\}$ is a basis of U.

$\dim U = m$.

$m+1$ vectors in U must be linearly dependent, so $(p_0, p_1, \cdots p_m)$ is linearly dependent.

PWE. 7. Prove by contradiction.

If $U \cap V = \{0\}$, Let $W = U \oplus V$,

$W \subseteq R^9$, but $\dim W = \dim U + \dim V = 5 + 5 = 10$.

Contradiction!

Chapter 6 PWE 1

Let dim(U) = m and dim(V) = n. $m \leq n$ since U is a subspace of V. Take a basis of U and let it be $\{u_1, u_2, ..., u_m\}$. According to the Basis extension theorem, one can extend it to get a basis of V, which is $\{u_1, u_2, ..., u_m, v_{m+1}, ..., v_n\}$. Define the linear transformation on V such that, for $v \in V = c_1u_1 + ... + c_mu_m + c_{m+1}v_{m+1} + ... + c_nv_n$,

$$T(v) = c_1S(u_1) + ... + c_mS(u_m).$$

- First, $T = S$ on U.
- Second, T is linear. For any $v_1 \in V = a_1u_1 + ... + a_mu_m + a_{m+1}v_{m+1} + ... + a_nv_n$ and $v_2 \in V = b_1u_1 + ... + b_mu_m + b_{m+1}v_{m+1} + ... + b_nv_n$,

  $$T(v_1 + v_2) = (a_1 + b_1)S(u_1) + ... + (a_m + b_m)S(u_m),$$

  and

  $$T(v_1) = a_1S(u_1) + ... + a_mS(u_m),$$
  $$T(v_2) = b_1S(u_1) + ... + b_mS(u_m).$$

  Hence, $T(v_1 + v_2) = T(v_1) + T(v_2)$. On the other hand,

  $$T(\lambda v_1) = (\lambda a_1)S(u_1) + ... + (\lambda a_m)S(u_m) = \lambda T(v_1).$$

Chapter 5 PWE. 2.

Take $v \in V$, there exists $\{c_1 \cdots c_n\}$ s.t.

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n.$$

Want to prove there exists $\{b_1, \cdots b_n\}$ s.t.

$$v = b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + \cdots + b_{n-1}(v_{n-1} - v_n) + b_nv_n.$$

$$= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + \cdots + (b_{n-1} - b_{n-2})v_{n-1} + (b_n - b_{n-1})v_n.$$

Let

$$\begin{aligned} b_1 &= c_1 \\ -b_1 + b_2 &= c_2 \\ -b_2 + b_3 &= c_3 \\ &\vdots \\ -b_{n-1} + b_n &= c_n \end{aligned} \Rightarrow \begin{cases} b_1 = c_1 \\ b_2 = c_1 + c_2 \\ b_3 = c_1 + c_2 + c_3 \\ \vdots \\ b_n = c_1 + c_2 + c_3 + \cdots + c_n. \end{cases}$$