Chapter 8.

CWE 1.

$$A = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -1 \end{bmatrix}$$

Expand along the frist row.

$$\det(A) = \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i \det\begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}$$

$$= -1 + i(i)$$

$$= -2.$$

$$\det(A^4) = (\det A)^4 = (-2)^4 = 16.$$

CWE 2.

(a). S_3:

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ no inversion even.

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. inversion $= 1$. odd.

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ 1 odd

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. 2. even.

$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ 2. even

$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. 3. odd.

(b). $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = \sum_{\pi \in S_3} \operatorname{sign}(\pi) \cdot a_{1\pi(1)} a_{2,\pi(2)} \cdot a_{3,\pi(3)}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$

$$+ a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}$$

CWE 3 : Similar to CWE2.

CWE 4. $\det\begin{bmatrix} x & -1 \\ 3 & 1-x \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{bmatrix}$

Left $= x(1-x)+3 = -x^2+x+3$

Right $= \begin{vmatrix} x & -6 \\ 3 & x-5 \end{vmatrix} - 3\begin{vmatrix} 2 & x \\ 1 & 3 \end{vmatrix} = x(x-5)+18-3(6-x)$

(Expand along the first row)

$$= x^2-5x+18-18+3x$$

$$= x^2-2x$$

Then $-x^2+x+3 = x^2-2x$

$$2x^2-3x-3=0$$

$$x = \frac{3\pm\sqrt{33}}{4}$$

CWE 5.

$$A = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{bmatrix}$$

Expand along the third column.

$$\det(A) = \det\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} = \sin^2\theta + \cos^2\theta = 1.$$

CWE 6.

First row + Second row − third row = [0 0 0].

Three rows are linearly dependent, so the matrix is not invertible.

PWE 1.

Want to prove $AB = BA \iff \det\begin{bmatrix} b & a-c \\ e & d-f \end{bmatrix} = 0$.

$$AB = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}$$

$$BA = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ad & bd+ce \\ 0 & cf \end{bmatrix}$$

$AB = BA \iff ae+bf = bd+ce$

$$\det\begin{bmatrix} b & a-c \\ e & d-f \end{bmatrix} = b(d-f) - e(a-c)$$

$$= bd - bf - ae + ce.$$

$$= 0$$

$\iff ae + bf = bd + ce.$

PWE 2

A is invertible. $\Leftrightarrow$ $\det(A) \neq 0$.

ATA is invertible $\Leftrightarrow$ $\det(A^TA) \neq 0$.

However, $\det(ATA) = \det(A^T)\det(A) = \det(A)\det(A)$

therefore, $\det(A) \neq 0$ if and only if $\det(ATA) \neq 0$.

PWE 3. False.

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\det(A) = 1$.

Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\det(B) = 1$.

$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\det(A+B) = 3$.

$\therefore \det(A+B) \neq \det(A) + \det(B)$

PWE4. False.

Let $r=2$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A) = 1$

$\det(rA) = \det\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$.

$\det(rA) \neq r\det(A)$.

The true statement is $\det(rA) = r^n \det(A)$.