Chapter 9. (Nine).

CE1.

$$f_1 = e_1 + e_2 + e_3.$$
$$f_2 = e_2 + e_3.$$
$$f_3 = e_3.$$

(a). Gram - Schmidt.

$$v_1 = \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{1+1+1}}(e_1+e_2+e_3) = \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 + \frac{1}{\sqrt{3}}e_3$$

$$v_2 = \frac{f_2 - \langle f_2, v_1\rangle v_1}{\|f_2 - \langle f_2, v_1\rangle v_1\|} = \frac{e_2+e_3 - \left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}e_1+\frac{1}{\sqrt{3}}e_2+\frac{1}{\sqrt{3}}e_3\right)}{\|f_2 - \langle f_2, v_1\rangle v_1\|}$$

$$= \frac{-\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3}{\|-\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3\|} = \sqrt{\frac{3}{2}}\left(-\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3\right).$$

$$= -\frac{\sqrt{2}}{\sqrt{3}}e_1 + \frac{1}{\sqrt{6}}e_2 + \frac{1}{\sqrt{6}}e_3.$$

$$v_3 = \frac{f_3 - \langle f_3, v_1\rangle v_1 - \langle f_3, v_2\rangle v_2}{\|f_3 - \langle f_3, v_1\rangle v_1 - \langle f_3, v_2\rangle v_2\|}$$

$$= \frac{e_3 - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}e_1+\frac{1}{\sqrt{3}}e_2+\frac{1}{\sqrt{3}}e_3\right) - \frac{1}{\sqrt{6}}\left(-\frac{\sqrt{2}}{\sqrt{3}}e_1+\frac{1}{\sqrt{6}}e_2+\frac{1}{\sqrt{6}}e_3\right)}{\|f_3 - \langle f_3, v_1\rangle v_1 - \langle f_3, v_2\rangle v_2\|}$$

$$= \frac{-\frac{5}{6}e_2 + \frac{5}{6}e_3}{\|-\frac{5}{6}e_2 + \frac{5}{6}e_3\|} = \sqrt{\frac{36}{50}}\left(-\frac{5}{6}e_2 + \frac{5}{6}e_3\right) = \frac{6}{5\sqrt{2}}\left(-\frac{5}{6}e_2 + \frac{5}{6}e_3\right).$$

$$= -\frac{1}{\sqrt{2}}e_2 + \frac{1}{\sqrt{2}}e_3.$$

(b).

$$w_1 = \frac{f_3}{\|f_3\|} = e_3.$$

$$w_2 = \frac{f_2 - \langle f_2, w_1\rangle w_1}{\|f_2 - \langle f_2, w_1\rangle w_1\|} = \frac{e_2+e_3-e_3}{\|e_2+e_3-e_3\|} = e_2.$$

$$w_3 = \text{— — ——— — — — —} = e_1$$

CE2.

$$\left\langle \frac{\sin jx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin jx \sin kx}{\pi}\, dx$$

$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(j-k)x - \cos(j+k)x]\, dx.$$

$$= \begin{cases} 1 & \text{if } j=k. \\ 0 & \text{if } j\neq k. \end{cases}$$

$$\left\langle \frac{\cos jx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\cos jx \cos kx}{\pi}\, dx$$

$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(j-k)x + \cos(j+k)x]\, dx$$

$$= \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j\neq k. \end{cases}$$

$$\left\langle \frac{\sin jx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin jx \cos kx}{\pi}\, dx$$

$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sin(j+k)x + \sin(j-k)x]\, dx$$

$$= 0.$$

CE 3. Let $f_1 = 1 \quad f_2 = x \quad f_3 = x^2$.

$$v_1 = \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{\int_0^1 1\,dx}} = 1.$$

$$v_2 = \frac{f_2 - \langle f_2, v_1\rangle v_1}{\|f_2 - \langle f_2, v_1\rangle v_1\|} = \frac{x - \left(\int_0^1 x\,dx\right)\cdot 1}{\|f_2 - \langle f_2, v_1\rangle v_1\|}$$

$$= \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 (x-\frac{1}{2})^2 dx}} = \frac{x - \frac{1}{2}}{\sqrt{1/12}} = 2\sqrt{3}x - \sqrt{3}.$$

$$v_3 = \frac{f_3 - \langle f_3, v_1\rangle v_1 - \langle f_3, v_2\rangle v_2}{\|f_3 - \langle f_3, v_1\rangle v_1 - \langle f_3, v_2\rangle v_2\|}$$

$$= \frac{x^2 - \left(\int_0^1 x^2 dx\right)\cdot 1 - \left(\int_0^1 x^2 (2\sqrt{3}x - \sqrt{3})\,dx\right)(2\sqrt{3}x - \sqrt{3})}{\|f_3 - \langle f_3, v_1\rangle v_1 - \langle f_3, v_2\rangle v_2\|}$$

$$\cancel{= \frac{x^2 - \frac{1}{3} - (12x-6)}{\|x^2 - \frac{1}{3} - (12x-6)\|} = \frac{x^2 - 12x + \frac{17}{3}}{\|x^2 - 12x - \frac{17}{3}\|}}$$

$$\cancel{= \frac{x^2 - 12x + \frac{17}{3}}{\int_0^1 (x^2 - 12x - \frac{17}{3})^2 dx} = \ldots\ldots}$$

$$= \frac{x^2 - \frac{1}{3} - \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3}\right)(2\sqrt{3}x - \sqrt{3})}{\|f_3 - \langle f_3, v_1\rangle v_1 - \langle f_3, v_2\rangle v_2\|}$$

$$= \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx}} = \sqrt{180}\,(x^2 - x + \tfrac{1}{6}) = 6\sqrt{5}\,(x^2 - x + \tfrac{1}{6})$$

CE.4.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,2,1)}{\|(1,2,1)\|} = \frac{(1,2,1)}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}(1,2,1) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

$$u_2 = \frac{v_2 - \langle v_2, u_1\rangle u_1}{\|v_2 - \langle v_2, u_1\rangle u_1\|}.$$

$$= \frac{(1,-2,1) - \left(-\frac{2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}{\|v_2 - \langle v_2, u_1\rangle u_1\|}$$

$$= \frac{\left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right)}{\left\|\left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right)\right\|} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

$$u_3 = \frac{v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2}{\|v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2\|}.$$

$$= \frac{(1,2,-1) - \left(\frac{4}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) - \left(-\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}{\|v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2\|}$$

$$= \frac{(1,0,-1)}{\|(1,0,-1)\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).$$

PWE 3.

Use Parallelogram Law Thm 9.3.6. Let $u=(x_1, y_1)$ $v=(x_2, y_2)$.

$$\| u+v \|^2 = \| (x_1+x_2 , y_1+y_2) \|^2$$
$$= (|x_1+x_2| + |y_1+y_2|)^2$$

$$\| u-v \|^2 = \| (x_1-x_2 , y_1-y_2) \|^2$$
$$= (|x_1-x_2| + |y_1-y_2|)^2$$

$$2(\|u\|^2 + \|v\|^2) = 2\left[(|x_1|+|y_1|)^2 + (|x_2| + |y_2|)^2\right]$$

Clearly $\|u+v\|^2 + \|u-v\|^2$ does not always equal $2(\|u\|^2+\|v\|^2)$.

For example, take $u=(1,-2)$, $v=(-2,1)$.

$$\|u+v\|^2 = (1+1)^2 = 4$$

$$\|u-v\|^2 = (3+3)^2 = 36.$$

$$2(\|u\|^2 + \|v\|^2) = 2(3^2+3^2) = 36.$$

$\Rightarrow$ $4+36 \neq 36$.

$\Rightarrow$ Therefore, this norm is not induced from an inner product.