Lecture #11

Study GROUPS: MSB 3118 TUES 1pm, Wed 4:30pm

A span Lemma: Let $w \in \text{span}(v_1, \dots, v_m)$. Then

$$\text{span}(v_1, \dots, v_m) = \text{span}(v_1, \dots, v_m, w)$$

PROOF $\boxed{\text{EXERCISE}}$

$\boxed{\text{EXERCISE}}$ Use this lemma to give a prettier proof for q1 on Midterm than Captain Conundrum!

LINEAR INDEPENDENCE

Def$^{\underline{n}}$ $v_1, \dots, v_m$ are called linearly independent if only sol$^{\underline{n}}$ to

$$0 = \alpha_1 v_1 + \dots + \alpha_m v_m$$

is $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$. I.e. this is the unique linear combination of $v_1, \dots, v_m$ equaling 0.

EX 4D TIC TAC TOE: $\alpha_1(1,0,0,0) + \alpha_2(0,1,0,0) + \alpha_3(0,0,1,0) + \alpha_4(0,0,0,1) = 0$ only has sol$^{\underline{n}}$

$$\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$$

b/c LHS $= (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ & $0 = (0,0,0,0) \Rightarrow (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ are linearly independent

PROPOSITION: The vectors $e_1, \dots, e_n \in \mathbb{F}^n$, where $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ with all but the i^{th} entry vanishing are linearly independent.

Proof: $\boxed{\text{EXERCISE}}$

Remark: If $v_1, \dots, v_m$ are <u>not</u> linearly independent we call them <u>linearly dependent</u>. This means that $\exists\ \alpha_1, \dots, \alpha_m \in \mathbb{F}$ <u>not all vanishing</u> such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

EX In $\mathbb{R}^2$ with $\rightarrow := (1,0)$, $\nearrow := (1,1)$ & $\uparrow := (0,1)$ we have

$$1 \cdot \rightarrow + (-1) \nearrow + 1 \cdot \uparrow = 0$$

so $\rightarrow$, $\nearrow$ & $\rightarrow$ are <u>linearly</u> <u>dependent</u>

Remark: Testing linear independence amounts to a linear systems problem

$\boxed{\text{EXERCISE}}$ Watch the Gaussian elimation video in ch 1, PWE 2. Explain how to use an augmented matrix & row operations to determine if a set of vectors in $\mathbb{F}^n$ are linearly independent. Give a simple example illustrating your methods.

LEMMA: $v_1, \ldots, v_m$ linearly independent

$\Leftrightarrow$ every $v \in \operatorname{span}(v_1, \ldots, v_m)$ can be uniquely expressed as a linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

PROOF ($\Rightarrow$): By contradiction: assume

$$v = \sum \alpha_i v_i \quad \& \quad v = \sum \alpha_i' v_i \quad \text{not all } \alpha_i = \alpha_i'.$$

Then $0 = v - v = \sum (\alpha_i - \alpha_i') v_i$, not all $\alpha_i - \alpha_i' = 0$

$\Rightarrow v_1, \ldots, v_m$ are linearly dependent $\lightning$

($\Leftarrow$): $0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_m$, but by assumption this expression is unique, thus $v_1, \ldots, v_m$ are linearly independent. $\square$

LINEAR DEPENDENCE LEMMA: Suppose $v_1, \ldots, v_m$ are linearly dependent & $v_1 \neq 0$.

Then $\exists\, j \in \{2, \ldots, n\}$ such that

i) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$

ii) $\operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \operatorname{span}(v_1, v_2, \ldots, v_m)$

REMARK This says that at least one vector is redundant (it can be written as a linear combination of the others & thrown out).

PROOF i) Since $v_1, \dots, v_m$ are linearly dependent we may write

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0$$

for some $\alpha_i \in \mathbb{F}$ not all zero. Since $v_1 \neq 0$ we cannot have $\alpha_2 = \dots = \alpha_n = 0$ (since then $\alpha_1 \neq 0 \Rightarrow v_1 = 0$ ↯). Pick the α_j with j such that all higher $\alpha_{j+1}, \dots$ vanish (this may require $j=m$). Then

$$v_j = -\frac{1}{\alpha_j}\left(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{j-1} v_{j-1}\right)$$

$$\Rightarrow v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$$

ii) Now let $v \in \operatorname{span}(v_1, \dots, v_m)$, then

$$v = \beta_1 v_1 + \dots + \beta_j v_j + \dots + \beta_m v_m = \left(\beta_1 - \frac{\alpha_1}{\alpha_j}\right) v_1 + \dots + \left(\beta_{j-1} - \frac{\alpha_{j-1}}{\alpha_j}\right) v_{j-1} + \beta_{j+1} v_{j+1} + \dots + \beta_m v_m$$

$$\in \operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$$

But v was chosen arbitrarily so $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$

EX $(\rightarrow, \nearrow, \uparrow) \in \mathbb{F}^2$ so we can drop one of these $\uparrow = \nearrow - \uparrow$ so $(\rightarrow, \nearrow)$ are linearly independent. Note we could reorder our list & get 2 other answers here!

THEOREM: Let $V = \text{span}(u_1, \dots, u_m)$ and $v_1, \dots, v_n \in V$ be linearly independent. Then if $V = \text{span}(v_1, \dots, v_n)$ also, then

$$n \leq m.$$

PROOF: $V = \text{span}(v_1, u_1, \dots, u_m)$ but $v_1, u_1, \dots, u_m$ are certainly linearly dependent, so by linear dependence lemma can drop u_j for some j. Now iterate this procedure, at any given step where $V = \text{span}(v_k, \dots, v_1, u_{k_1}, \dots, u_{k_{m-k}})$ when we add an additional v_{k+1} we must then drop one of $u_{k_1}, \dots, u_{k_{m-k}}$ because $v_1, \dots, v_k$ are linearly independent.

The endpoint of this iteration is $V = \text{span}(v_1, \dots, v_n, \mathcal{U})$ for some subset $\mathcal{U} \subset \{u_1, \dots, u_m\}$. Hence $n \leq m$.

Remark: We would like to use n as the dimension of V, but perhaps it is not unique.

NEXT **BASES**