Lecture #12

LINEAR DEPENDENCE LEMMA: Suppose $v_1, \dots, v_m$ are linearly dependent & $v_1 \neq 0$.
Then $\exists\, j \in \{2, \dots, n\}$ such that

i) $v_j \in \text{span}(v_1, \dots, v_{j-1})$

ii) $\text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) = \text{span}(v_1, v_2, \dots, v_m)$

REMARK This says that at least one vector is redundant (it can be written as a linear combination of the others & thrown out).

PROOF i) Since $v_1, \dots, v_m$ are linearly dependent we may write

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0$$

for some $\alpha_i \in F$ not all zero. Since $v_1 \neq 0$ we cannot have $\alpha_2 = \dots = \alpha_n = 0$ (since then $\alpha_1 \neq 0 \Rightarrow v_1 = 0$ ↯). Pick the α_j with j such that all higher $\alpha_{j+1}, \dots$ vanish (this may require $j = m$). Then

$$v_j = -\frac{1}{\alpha_j}\left(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{j-1} v_{j-1}\right)$$

$$\Rightarrow v_j \in \text{span}(v_1, \dots, v_{j-1})$$

ii) Now let $v \in \text{span}(v_1, \dots, v_m)$, then

$$v = \beta_1 v_1 + \dots + \beta_j v_j + \dots + \beta_m v_m = \left(\beta_1 - \frac{\alpha_1}{\alpha_j}\right) v_1 + \dots + \left(\beta_{j-1} - \frac{\alpha_{j-1}}{\alpha_j}\right) v_{j-1} + \beta_{j+1} v_{j+1} + \dots + \beta_m v_m$$

$$\in \text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$$

But v was chosen arbitrarily so $\text{span}(v_1, \dots, v_m) = \text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$

<u>EX</u> $(\rightarrow, \nearrow, \uparrow) \in \mathbb{F}^2$ so we can drop one of these $\uparrow = \nearrow - \uparrow$ so $(\rightarrow, \nearrow)$ are linearly independent. <u>Note</u> we could reorder our list & get 2 other answers here!

<u>THEOREM:</u> Let $V = \text{span}(u_1, \dots, u_m)$ and $v_1, \dots, v_n \in V$ be linearly independent. Then if $V = \text{span}(v_1, \dots, v_n)$ also, then

$$n \leq m.$$

<u>PROOF:</u> $V = \text{span}(v_1, u_1, \dots, u_m)$ but $v_1, u_1, \dots, u_m$ are certainly linearly dependent, so by linear dependence lemma can drop u_j for some j. Now iterate this procedure, at any given step where $V = \text{span}(v_k, \dots, v_1, u_{k_1}, \dots, u_{k_{m-k}})$ when we add an additional v_{k+1} we must then drop one of $u_{k_1}, \dots, u_{k_{m-k}}$ because $v_1, \dots, v_k$ are linearly independent.

The endpoint of this iteration is $V = \text{span}(v_1, \dots, v_n, \mathcal{U})$ for some subset $\mathcal{U} \subset \{u_1, \dots, u_m\}$. Hence $n \leq m$.

<u>Remark</u>: We would like to use n as the dimension of V, but perhaps it is not unique.

Bases

Def$\underline{n}$ Call $v_1, \dots, v_m$ a basis for V if

i) $v_1, \dots, v_m$ are linear independent

ii) $\mathrm{span}(v_1, \dots, v_m) = V$

Remarks

- We call the number $m = \dim V$. Later we show it is the same for all bases of V, so it is a good invariant of V.
- The above assumes we can find a list of finitely many vectors that span V. In this case we call V finite dimensional
- In fact all finite dimensional vector spaces are isomorphic to $\mathbb{F}^m$; this is good news (but also a let down ...).

Note To establish an "isomorphism" b/w V with basis $v_1, \dots, v_m$ & $\mathbb{F}^m$ just write

$$V \ni v = \underbrace{\alpha_1 v_1 + \dots + \alpha_m v_m}_{\text{unique}} \longleftrightarrow (\alpha_1, \dots, \alpha_m) \in \mathbb{F}^m$$

more later...

Basis Reduction Theorem: If $V = \text{span}(v_1, \dots, v_m)$ either $v_1, \dots, v_m$ is a basis for V or some subset of $\{v_1, \dots, v_m\}$ can be removed to obtain a basis.

Proof If $v_1, \dots, v_m$ are not linearly independent use the Linear dependence Lemma to remove some v_j (if $v_1 = 0$ first remove it). Iterate until linearly independent

EX $\{\rightarrow, \nearrow, \uparrow\} \longrightarrow \{\rightarrow, \nearrow\}$ or $\{\rightarrow, \uparrow\}$ or $\{\nearrow, \uparrow\}$

Corollary Every finite dimensional vector space has a basis.

Basis Extension Theorem: If V is finite dimensional, any linearly independent set of vectors $\{u_1, \dots, u_k\}$ can be extended to a basis.

Proof Since V is finite dimension it certainly has a basis $v_1, \dots, v_m$ so now consider the list $(u_1, \dots, u_k, v_1, \dots, v_m)$ which is certainly linearly dependent. Now use Linear dependence Lemma to succesively remove v_j for some j (this can never remove u_j's b/c these are linearly independent. The result $\{u_1, \dots, u_k, \mathcal{V}\}$ with $\mathcal{V} \subset \{v_1, \dots, v_m\}$ will be linearly independent & span V $\square$

EX $\{\nearrow\} \longrightarrow \{\nearrow, \rightarrow, \uparrow\} \longrightarrow \{\nearrow, \rightarrow\}$

DIMENSION

Theorem: Let $v_1, \dots, v_n$ & $u_1, \dots, u_m$ be bases for V.

Then $n = m$

Proof: We proved already that $V = \text{span}(u_1, \dots, u_m)$ & $v_1, \dots, v_n$ linearly independent, then $V = \text{span}(v_1, \dots, v_n) \Rightarrow n \leq m$ so $n \leq m$. Swapping u's & v's gives $m \leq n \Rightarrow n = m$

Remarks

- The number of vectors in any basis of a finite dimensional vectorspace is called $\dim V$.
- If $\dim V$ vectors are linearly independent then they form a basis.
- If U is a subspace of V, $\dim U \leq \dim V$
- If $\dim V$ vectors span V then they form a basis

Observe, if W is a subspace of V a finite dimensional vector space, it has a basis $w_1, \dots, w_m$ $\quad m = \dim V$.

But $w_1, \dots, w_m$ are linearly independant in W, so also in V and thus can be extended to a basis $\{w_1, \dots, w_m, v_1, \dots, v_{\dim V - \dim W}\}$.

Then

$$V = W \oplus \operatorname{span}(v_1, \dots, v_{\dim V - \dim W})$$

This is a direct sum because if

$$v \in W \cap \operatorname{span}(v_1, \dots, v_{\dim V - \dim W})$$

then $v = \alpha_1 w_1 + \dots + \alpha_m w_m = \beta_1 v_1 + \dots + \beta_{\dim V - \dim W} v_{\dim V - \dim W}$

$\Rightarrow 0 = \alpha_1 w_1 + \dots + \alpha_m w_m + (-\beta_1) v_1 + \dots + (-\beta_{\dim V - \dim W}) v_{\dim V - \dim W}$

$\Rightarrow (w_1, \dots, w_m, v_1, \dots, v_{\dim V - \dim W})$ is not a basis $\lightning$

We have proven the following

THEOREM If V is finite dimensional and $W \subset V$ is a subspace, then $\exists$ a subspace U such that $V = W \oplus U$.

Moreover $\dim V = \dim W + \dim U$

$\boxed{\text{EXERCISE}}$ Generalize the above formula to the case

$V = W + U$ the subspace sum.