Lecture #12

BASES & DIMENSION

SUMMARY

* V finite dimensional $\Leftrightarrow$ $V = \mathrm{span}(v_1, \dots, v_m)$

* $v_1, \dots, v_m$ span & linearly independent

$$\Updownarrow$$

$v_1, \dots, v_m$ a basis & $m =: \dim V$

A basis is a good way to label elements of $V \ni v$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$$

with $(\alpha_1, \dots, \alpha_m) \in \mathbb{F}^m$, <u>unique</u> for each v.

Key tool: Linear dependence Lemma.

Last time

$$V = \text{span}(\underbrace{u_1, \dots, u_m}_{m \text{ elements}}) \qquad v_1, \dots, v_n \text{ linearly independent}$$

insert a v, remove a u

NEVER EXHAUSTS v's BY LINEAR INDEPENDENCE

$$V = \text{span}(\underbrace{v_1, \dots, v_n, u}_{m \text{ elements}})$$

NUMBER OF LINEARLY INDEPENDENT VECTORS ALWAYS LESS THAN THAT OF SPANNING SET FOR V.

Goal Show m is independent of choice of basis.

First ingredient:

Basis Reduction Theorem: If $V = \operatorname{span}(v_1, \dots, v_m)$ either $v_1, \dots, v_m$ is a basis for V or some subset of $\{v_1, \dots, v_m\}$ can be removed to obtain a basis.

Proof If $v_1, \dots, v_m$ are **not** linearly independent use the Linear dependence Lemma to remove some v_j (if $v_1 = 0$ first remove it). **Iterate** until linearly independent

EX $\{\rightarrow, \nearrow, \uparrow\} \longrightarrow \{\rightarrow, \nearrow\}$ or $\{\rightarrow, \uparrow\}$ or $\{\nearrow, \uparrow\}$

Corollary Every finite dimensional vector space has a basis.

__Basis Extension Theorem__: If V is finite dimensional, any linearly independent set of vectors $\{u_1, \dots, u_k\}$ can be extended to a basis.

__Proof__ Since V is finite dimension it certainly has a basis $v_1, \dots, v_m$ so now consider the list $(u_1, \dots, u_k, v_1, \dots, v_m)$ which is certainly linearly dependent. Now use Linear dependence Lemma to succesively remove v_j for some j [this can never remove u_j's b/c these are linearly independent. The result $\{u_1, \dots, u_k, \mathcal{V}\}$ with $\mathcal{V} \subset \{v_1, \dots, v_m\}$ will be linearly independent & span V $\square$

__EX__ $\{\nearrow\} \longrightarrow \{\nearrow, \rightarrow, \uparrow\} \longrightarrow \{\nearrow, \rightarrow\}$

DIMENSION

Theorem: Let $v_1, \dots, v_n$ & $u_1, \dots, u_m$ be bases for V.

Then $n = m$

Proof: We proved already that $V = \text{span}(u_1, \dots, u_m)$ & $v_1, \dots, v_n$ linearly independent, then $V = \text{span}(v_1, \dots, v_n) \Rightarrow n \leq m$ so $n \leq m$. Swapping u's & v's gives $m \leq n \Rightarrow n = m$

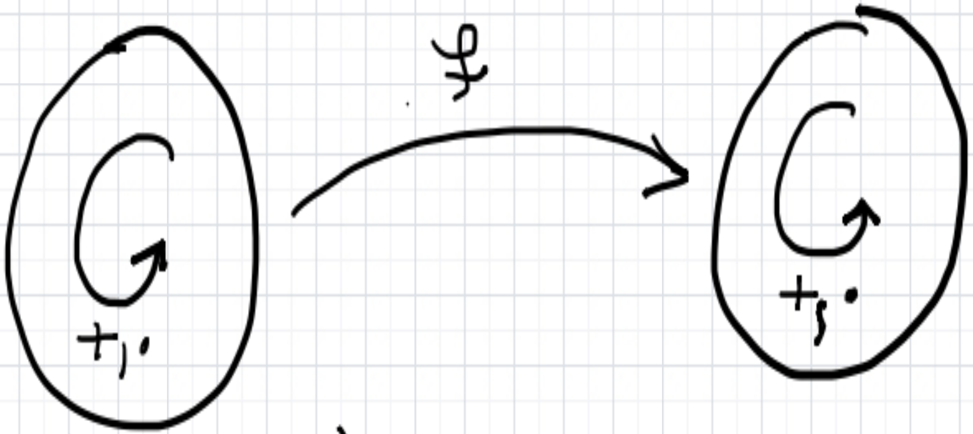
Remarks

- The number of vectors in any basis of a finite dimensional vectorspace is called $\dim V$.
- If $\dim V$ vectors are linearly independent then they form a basis.
- If U is a subspace of V, $\dim U \leq \dim V$
- If $\dim V$ vectors span V then they form a basis

$\boxed{\text{EXERCISES}}$ (i) Make sure you can prove the above four remarks.
(ii) Read & understand the applications of the basis extension theorem to subspaces Thm 5.4.5 & 5.4.6

LINEAR TRANSFORMATIONS (LINEAR MAPS) Ch. 6

Alias VECTOR SPACE HOMOMORPHISM



$(\mathcal{U}, +_{\mathcal{U}}, \mathbb{F}, \cdot_{\mathcal{U}})$ $\qquad\qquad$ $(V, +_V, \mathbb{F}, \cdot_V)$

$$f(u +_{\mathcal{U}} w) = f(u) +_V f(w)$$

$$f(\lambda \cdot_{\mathcal{U}} u) = \lambda \cdot_V f(u)$$

Commutative diagrams

$$\begin{tikzcd} U \times U \arrow[r, "f \times f"] \arrow[d, "+"'] \arrow[dr, phantom, "\circlearrowleft"] & V \times V \arrow[d, "+"] \\ U \arrow[r, "f"'] & V \end{tikzcd} \qquad \begin{tikzcd} U \times \mathbb{F} \arrow[r, "f \times \mathrm{Id}"] \arrow[d, "\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & U \times \mathbb{F} \arrow[d, "\cdot"] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$