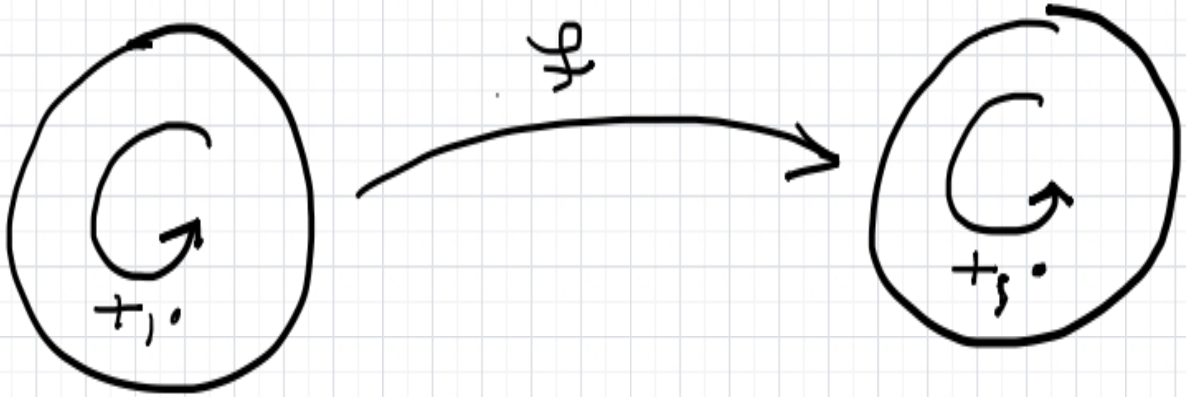
Lecture #14 – WORKSHEET – TEX TEMPLATES – BLANK MIDTERMS

LINEAR TRANSFORMATIONS (LINEAR MAPS) Ch. 6

Alias VECTOR SPACE HOMOMORPHISM



$(U, +_U, \mathbb{F}, \cdot_U)$ $\qquad$ $(V, +_V, \mathbb{F}, \cdot_V)$

$$f(u +_U w) = f(u) +_V f(w)$$

$$f(\lambda \cdot_U u) = \lambda \cdot_V f(u)$$

In modern language ...

Commutative diagrams

$$\begin{tikzcd} U \times U \arrow[r, "f \times f"] \arrow[d, "+"'] \arrow[dr, phantom, "\circlearrowleft"] & V \times V \arrow[d, "+"] \\ U \arrow[r, "f"'] & V \end{tikzcd} \qquad \begin{tikzcd} U \times \mathbb{F} \arrow[r, "f \times \mathrm{Id}"] \arrow[d, "\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & U \times \mathbb{F} \arrow[d, "\cdot"] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

You can add linear maps:

If $f, g : V \xrightarrow{\text{linear}} W$

then
$$\begin{array}{ccc} (f+g) : V & \longrightarrow & W \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & f(v)+g(v) \end{array}$$

$$\begin{array}{ccc} (\lambda \cdot f) : V & \longrightarrow & W \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & \lambda f(v) \end{array}$$

are also linear maps $V \xrightarrow{\text{linear}} W$

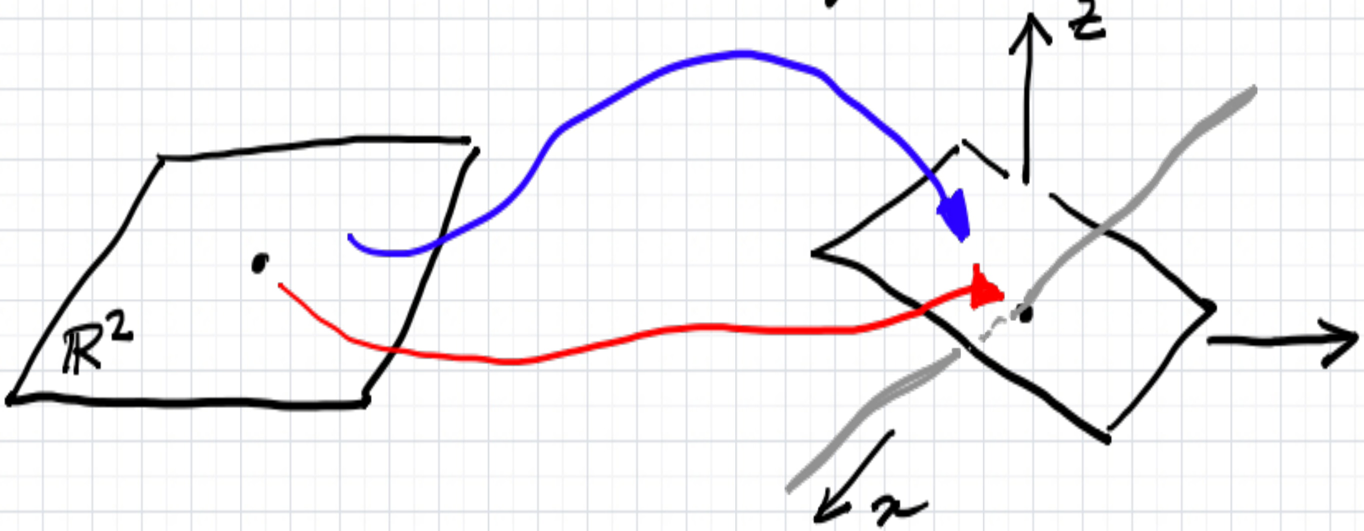
Call the set of all linear maps $V \xrightarrow{\text{linear}} W$

$$\mathcal{L}(V, W)$$

Theorem $\mathcal{L}(V, W)$ is a vector space.

$\boxed{\text{EXERCISE}}$ Prove this. Hint – zero vector?

EX $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is space of maps



These are characterized by pairs of vectors in $\mathbb{R}^3$! Suspect

$$\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3) \text{“}\cong\text{”} \mathbb{R}^3 \times \mathbb{R}^3$$

Definition The space of linear maps

$$V \xrightarrow{\text{linear}} \mathbb{F}$$

where V is a vector space over $\mathbb{F}$ is called the dual space, denoted

$$V^* = \mathcal{L}(V, \mathbb{F})$$

Remark This is sometimes called the space of linear functionals.

EX $(\mathbb{Z}_3^2)^* =$ dual tic tac toe

claim this is isomorphic to TIC TAC TOE ITSELF!

$$= \{ f(x,y)=0,\ f(x,y)=x,\ f(x,y)=2x,\ f(x,y)=y,\ f(x,y)=x+y,$$
$$f(x,y)=2x+y,\ f(x,y)=2y,\ f(x,y)=x+2y,\ f(x,y)=2x+2y \}$$

Theorem Let $(v_1, \ldots, v_m)$ be a basis for V and $w_1, \ldots, w_m \in W$. Then there is a unique

$$T : V \xrightarrow{\text{linear}} W$$

such that $T(v_1) = w_1, \ldots, T(v_m) = w_m$.

Proof First we construct $T(v)$ by writing $v = \sum_{i=1}^{m} \alpha_i v_i$ and set

$$T(v) = \alpha_1 w_1 + \cdots + \alpha_m w_m .$$

Then $T(v+v') = \sum_i (\alpha_i + \alpha_i') w_i = \sum_i \alpha_i w_i + \sum_i \alpha_i' w_i = T(v) + T(v')$

similarly $T(\lambda v) = \sum_i \lambda \alpha_i w_i = \lambda \sum_i \alpha_i w_i = \lambda T(v)$

so T is linear. Now suppose $T'(v_1) = w_1, \ldots, T'(v_m) = w_m$, but $T(v) \neq T'(v)$ for some v and T' linear. Then

$$0 \neq T(v) - T'(v) = T\left(\sum_i \alpha_i v_i\right) - T'\left(\sum_i \alpha_i v_i\right) = \sum_i \alpha_i T(v_i) - \sum_i \alpha_i T'(v_i)$$

$$= \sum_i \alpha_i w_i - \sum_i \alpha_i w_i = 0$$

QED

The Matrix of a linear transformation: Let $m = \dim V$, $n = \dim W < \infty$ and $T : V \xrightarrow{\text{linear}} W$. Choose bases $(v_1, \dots, v_m)$, $(w_1, \dots, w_n)$.

Then we can write uniquely

$$T(v_1) = t_1^1 w_1 + t_1^2 w_2 + \dots + t_1^n w_n, \dots, T(v_m) = t_m^1 w_1 + t_m^2 w_2 + \dots + t_m^n w_n$$

$$:= (w_1 \ w_2 \cdots w_n) \begin{pmatrix} t_1^1 \\ t_1^2 \\ \vdots \\ t_1^n \end{pmatrix}, \dots, \quad := (w_1 \ w_2 \cdots w_n) \begin{pmatrix} t_m^1 \\ t_m^2 \\ \vdots \\ t_m^n \end{pmatrix}$$

$$\Rightarrow (T(v_1) \ T(v_2) \cdots T(v_m)) = (w_1 \ w_2 \cdots w_m) \begin{pmatrix} t_1^1 & t_2^1 & \cdots & t_m^1 \\ t_1^2 & t_2^2 & \cdots & t_m^2 \\ \vdots & & \ddots & \vdots \\ t_1^n & t_2^n & \cdots & t_m^n \end{pmatrix}$$

This $n \times m$ array of numbers is called the matrix of T

EXAMPLE $\quad f(x,y) = (ax+by,\ cx+dy)$

$$f : \mathbb{R}^2 \xrightarrow{\text{linear}} \mathbb{R}^2$$

Choose bases $\rightarrow, \uparrow$ in both $\mathbb{R}^2$'s.

$$f(\rightarrow) = (a, c) = a \rightarrow + c \uparrow, \quad f(\uparrow) = b \rightarrow + d \uparrow$$

$$\Rightarrow (f(\rightarrow)\ \ f(\uparrow)) = (\rightarrow\ \ \uparrow)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Notice $(x,y) = x \rightarrow + y \uparrow \longmapsto (ax+by) \rightarrow + (cx+dy) \uparrow$

$$= (\rightarrow\ \ \uparrow)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (\rightarrow\ \ \uparrow)\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$