Lecture 15 Linear Maps

Theorem Let $v_1, \ldots, v_n$ be a basis for V. Then given $w_1, \ldots, w_n \in W$ there is a unique <u>linear</u> map $f : V \to W$ s.t. $f(v_1) = w_1, \ldots, f(v_n) = w_n$.

Proof First construct f. Any $v \in V$ can be uniquely expressed

$$v = \alpha_1 v_1 + \ldots \alpha_n v_n$$

Set $f(v) = \alpha_1 w_1 + \ldots + \alpha_n v_n$. By construction $f(v_i) = w_i$.

Moreover $f(\mu v + \lambda v') = f((\mu\alpha_1 + \lambda\alpha_1')v_1 + \cdots + (\mu\alpha_n + \lambda\alpha_n')v_n)$

$$= (\mu\alpha_1 + \lambda\alpha_1')w_1 + \cdots + (\mu\alpha_n + \lambda\alpha_n')w_n = \mu(\alpha_1 w_1 + \cdots + \alpha_n w_n) + \lambda(\alpha_1' w_1 + \cdots \alpha_n' w_n)$$

$= \mu f(v) + \nu f(v') \implies f$ is linear.

Finally, uniqueness. Suppose $f(v) \neq f'(v)$ for some f & f' both linear and subject to $f(v_i) = w_i = f(v_i')$. Then $0 \neq f(v) - f(v')$

$$= f(\alpha_1 v_1 + \cdots + \alpha_n v_n) - f'(\alpha_1 v_1 + \cdots + \alpha_n v_n) \underset{\text{linear}}{=} \alpha_1 f(v_1) + \cdots + \alpha_n f(v_1) - (\alpha_1 f'(v_1) + \cdots + \alpha_n f'(v_n))$$

$$= \alpha_1 w_1 + \cdots + \alpha_n w_n - (\alpha_1 w_1 + \cdots + \alpha_n w_n) = 0$$

QED

The Matrix of a linear transformation: Let $m = \dim V$, $n = \dim W < \infty$

and $T : V \xrightarrow{\text{linear}} W$. Choose bases $(v_1, \dots, v_m)$, $(w_1, \dots, w_n)$.

Then we can write uniquely

$$T(v_1) = t_1^1 w_1 + t_1^2 w_2 + \cdots + t_1^n w_n, \dots, T(v_m) = t_m^1 w_1 + t_m^2 w_2 + \cdots + t_m^n w_n$$

$$:= (w_1 \ w_2 \cdots w_n) \begin{pmatrix} t_1^1 \\ t_1^2 \\ \vdots \\ t_1^n \end{pmatrix}, \dots, \quad := (w_1 \ w_2 \cdots w_n) \begin{pmatrix} t_m^1 \\ t_m^2 \\ \vdots \\ t_m^n \end{pmatrix}$$

$$\Rightarrow (T(v_1) \ T(v_2) \cdots T(v_m)) = (w_1 \ w_2 \cdots w_m) \begin{pmatrix} t_1^1 & t_2^1 & \cdots & t_m^1 \\ t_1^2 & t_2^2 & \cdots & t_m^2 \\ \vdots & & \ddots & \vdots \\ t_1^n & t_2^n & \cdots & t_m^n \end{pmatrix}$$

This $n \times m$ array of numbers is called the matrix of T

<u>Summary</u> $T(v_i) = \sum_{j=1}^{\dim W} t_i^j \Rightarrow M(T) = (t_i^j)$ ← rows $1, \dots, \dim W$ ← columns $1, \dots, \dim V$

EXAMPLE $\quad f(x,y) = (ax+by,\ cx+dy)$

$$f : \mathbb{R}^2 \xrightarrow{\text{linear}} \mathbb{R}^2$$

Choose bases $\rightarrow, \uparrow$ in both $\mathbb{R}^2$'s.

$$f(\rightarrow) = (a,c) = a\rightarrow + c\uparrow,\quad f(\uparrow) = b\rightarrow + d\uparrow$$

$$\Rightarrow \left(f(\rightarrow)\ \ f(\uparrow)\right) = (\rightarrow\ \ \uparrow)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Notice $(x,y) = x\rightarrow + y\uparrow \longmapsto (ax+by)\rightarrow + (cx+dy)\uparrow$

$$= (\rightarrow\ \ \uparrow)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (\rightarrow\ \ \uparrow)\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Composition of linear transformations & their algebra

$$V \xrightarrow{f} W \xrightarrow{g} U$$

Then

$$\begin{tikzcd}
& W \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowright"] & \\
V \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & {} & U
\end{tikzcd}$$

$v \in V$

where $(g \circ f)(v) = g(f(v))$

<u>Nb</u> often write $(g \circ f)(v) = gfv$ because composition of maps associates

$$h(g(f(v))) = (h \circ g)(f(v)) = h(g \circ f(v))$$

<u>Identity map:</u> $\mathrm{Id} : V \longrightarrow V$, $v \longmapsto v$ obeys $\mathrm{Id}\, f = f\, \mathrm{Id}$ (in W; in V)

<u>Distributivity:</u> $f \circ (\alpha g + \beta g') = \alpha f \circ g + \beta f \circ g'$, $(\alpha f + \beta f') \circ g = \alpha f \circ g + \beta f' \circ g$

Matrix Multiplication

If $f: V \longrightarrow W$ & $g: W \longrightarrow U$

where $\{v_1, \dots, v_m\}$, $\{w_1, \dots, w_n\}$, $\{u_1, \dots, u_r\}$ are respective bases, and

$$f(v_i) = \sum_j w_j t_i^j \quad ; \quad g(w_j) = \sum_k u_k s_j^k$$

so that $M(f) = (t_i^j)_{n \times m}$, $\quad M(g) = (s_j^k)_{r \times n}$

Then $(g \circ f)(v_i) = g\left(\sum_j w_j t_i^j\right) \overset{\text{LINEAR}}{=} \sum_j t_i^j g(w_j) = \sum_j t_i^j \sum_k u_k s_j^k = \sum_k u_k \left(\sum_j s_j^k t_i^j\right)$

Thus $M(g \circ f) = \underbrace{\left(\sum_j s_j^k t_i^j\right)}_{r \times m} =: M(g)M(f)$ MATRIX MULTIPLICATION

<u>Note</u> Most people abbreviate

$$M(f) = (t_i^j) = T$$

$$M(g) = (s_j^k) = S$$

<u>Thus</u> $M(g \circ f) = ST$ where $ST = (s_j^k t_i^j)$

in the notation of <u>Einstein</u> where repeated indices are summed.

So the matrix t_i^j encodes $f: V \to W$ so long as we know which basis we work with.

Similarly writing $v \in V$ as $v = \sum_i v_i \alpha^i$, the coefficients α^i encode the vector v.

Physicists would call t_i^j a <u>matrix</u>, α^i a <u>vector</u> and $t_i^j \alpha^i$ the <u>vector</u> $f(v)$ because $f(v) = f(v_i \alpha^i) = f(v_i)\alpha^i = w_j t_i^j \alpha^i$

The Euclidean Group: Consider the following maps

$$\left(\begin{array}{c|c} R & v \\ \hline & 1\end{array}\right)\left(\begin{array}{c|c} R^{-1} & \\ \hline & 1\end{array}\right) = \left(\begin{array}{c|c} 1 & v \\ \hline 0 & 1\end{array}\right)$$

$$\begin{array}{ccc} R_\theta : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \psi & & \psi \\ (x,y,z) & \longmapsto & (\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y, z) \end{array}$$

$$\begin{array}{ccc} T_{(a,b)} : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \psi & & \psi \\ (x,y,z) & \longrightarrow & (x+a, y+b, z) \end{array}$$

* $\boxed{\text{Compute}}$ their matrices in the bases $(1,0,0), (0,1,0), (0,0,1)$.
* $\boxed{\text{Calculate}}$ $R_\theta \circ R_\varphi$; what do you observe?
* $\boxed{\text{calculate}}$ $T_{(a,b)} \circ T_{(a',b')}$; now what do you observe?

Hint — use matrix multiplication.

* **Harder**: What about $R_\theta \circ T_{(a,b)} \circ R_{-\theta}$