Lecture #16 Linear maps

Algebra of linear maps

$$U \xrightarrow{g} V \xrightarrow{f} W$$

$$\begin{tikzcd}
 & V \arrow[dr, "f"] & \\
U \arrow[ur, "g"] \arrow[rr, "f \circ g"] & & W \\
u \arrow[rr, mapsto] & & f(g(u))
\end{tikzcd}$$

* composition $f \circ g$ is associative: $e \circ (f \circ g) = (e \circ f) \circ g$

  b/c both act via $e(f(g(u)))$.

  so write fg for $f \circ g$ $\rightsquigarrow$ multiplication

* Can also add linear transformations $f + f'$

* Plus there is an identity $\mathrm{Id}: V \longrightarrow V$ via $\mathrm{Id}(v) = v$

* Distributive $f(\alpha g + \beta g') = \alpha fg + \beta fg'$

  $(\alpha f + \beta f')g = \alpha fg + \beta f'g$

"Unital, associative algebra"

Matrix Multiplication

If $f: V \longrightarrow W$ & $g: W \longrightarrow U$

where $\{v_1, \dots, v_m\}$, $\{w_1, \dots, w_n\}$, $\{u_1, \dots, u_r\}$ are respective bases, and

$$f(v_i) = \sum_j w_j t_i^j \quad ; \quad g(w_j) = \sum_k u_k s_j^k$$

so that $M(f) = (t_i^j)\ n \times m$, $\quad M(g) = (s_j^k)\ r \times n$

Then $(g \circ f)(v_i) = g\left(\sum_j w_j t_i^j\right) \overset{\text{LINEAR}}{=} \sum_j t_i^j g(w_j) = \sum_j t_i^j \sum_k u_k s_j^k = \sum_k u_k \left(\sum_j s_j^k t_i^j\right)$

Thus $M(g \circ f) = \underbrace{\left(\sum_j s_j^k t_i^j\right)}_{r \times m} =: M(g) M(f)$ MATRIX MULTIPLICATION

The Euclidean Group: Consider the following maps

$$\begin{array}{rccc} R_\theta : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & \Psi & & \Psi \\ & (x,y,z) & \longmapsto & (\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y, z) \end{array}$$

$$\begin{array}{rccc} T_{(a,b)} : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & \Psi & & \Psi \\ & (x,y,z) & \longrightarrow & (x+a, y+b, z) \end{array}$$

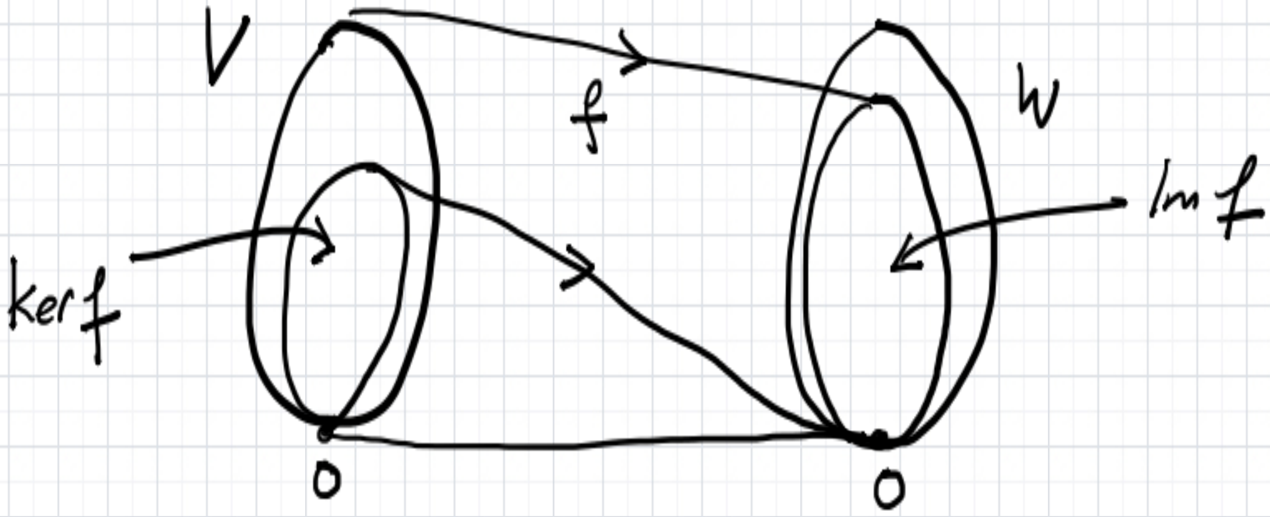
* $\boxed{\text{Compute}}$ their matrices in the bases $(1,0,0), (0,1,0), (0,0,1)$.
* $\boxed{\text{Calculate}}$ $R_\theta \circ R_\varphi$; what do you observe?
* $\boxed{\text{calculate}}$ $T_{(a,b)} \circ T_{(a',b')}$; now what do you observe?

Hint — use matrix multiplication.

* Harder: What about $R_\theta \circ T_{(a,b)} \circ R_{-\theta}$

Def$^{\underline{n}}$: Let $f: V \longrightarrow W$. Then

$$\ker f = \{v \mid f(v) = 0\} \subset V$$



$$\operatorname{Im} f = \{f(v) \mid v \in V\} \subset W$$

Theorem: $\ker f$ and $\operatorname{Im} f$ are subspaces of V & W, respectively.

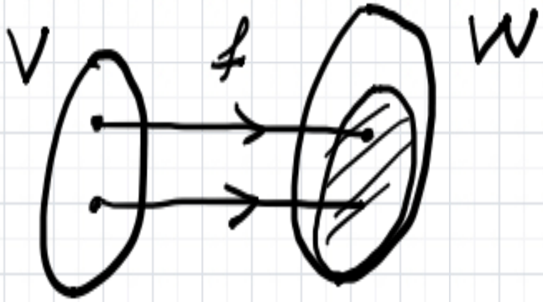
We proved this already

Remark: $\ker f \equiv \operatorname{null} f$. $\operatorname{Im} f \equiv f(V) \equiv \operatorname{ran} f$.

Definition: $f: V \to W$ is called <u>injective</u> if

$$f(v) = f(u) \Rightarrow u = v$$

Remark: This is sometimes called $1:1$



f is "injecting" a copy of of V into W

Theorem Let $f: V \xrightarrow[\text{linear}]{} W$. Then $\ker f = \{0\} \Leftrightarrow f$ injective.

Proof $(\Rightarrow)$ Suppose $\ker f = \{0\}$ and $f(u) = f(v)$.

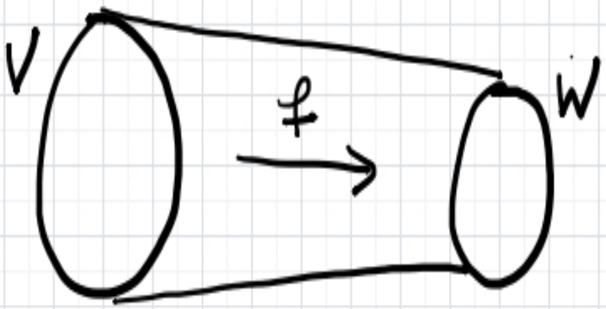
Then $0 = f(u) - f(v) \underset{\text{LINEAR}}{=} f(u-v) \Rightarrow u - v \in \ker f = \{0\} \Rightarrow u = v$

$(\Leftarrow)$ Suppose f is injective and $u \in \ker f$. Then $f(u) = 0$.

But $f(0) = f(0 \cdot 0) = 0 \cdot f(0) = 0 \Rightarrow f(u) = f(0) \Rightarrow u = 0$ QED

Ex $\frac{d}{dx}: \mathbb{F}[x] \to \mathbb{F}[x]$ is <u>not</u> $1:1$ b/c $p(x) = c \in \ker \frac{d}{dx}$.

Defⁿ: We say $f: V \to W$ is surjective if $\operatorname{Im} f = W$.

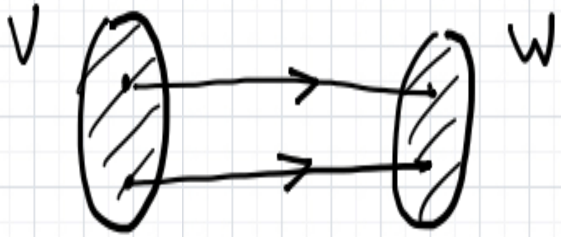


i.e. $\forall w \in W$, $\exists v \in V$ such that $f(v) = w$.

$*$ v need <u>not</u> be unique.

EX $\frac{d}{dx} : \mathbb{F}[x] \to \mathbb{F}[x]$ b/c $\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = \frac{d}{dx}\left(\alpha_0 x + \frac{\alpha_1}{2} x^2 + \cdots + \frac{\alpha_n}{n+1} x^n\right)$.

Defⁿ: We say $f : V \to W$ is a bijection if f is 1:1 and onto.



Remark: When $f : V \xrightarrow[\text{Linear}]{} W$ (homomorphism) we call

$$\begin{cases} \text{surjective} - \text{epimorphism} \\ \text{injective} - \text{monomorphism} \\ \text{bijective} - \text{isomorphism} \end{cases}$$

Defⁿ: If $f: V \to W$ is an isomorphism, we say V & W are <u>isomorphic</u>, $V \cong W$. In some sense they are the same vector space.

EX $\mathrm{Id} : V \to V$ is an isomorphism.

We call
$$\begin{tikzcd} V \arrow[loop above, "\text{linear } f"] \end{tikzcd}$$
an endomorphism.

Bijective endomorphisms are called automorphisms.

<u>Next</u>: The dimension formula & invertibility.