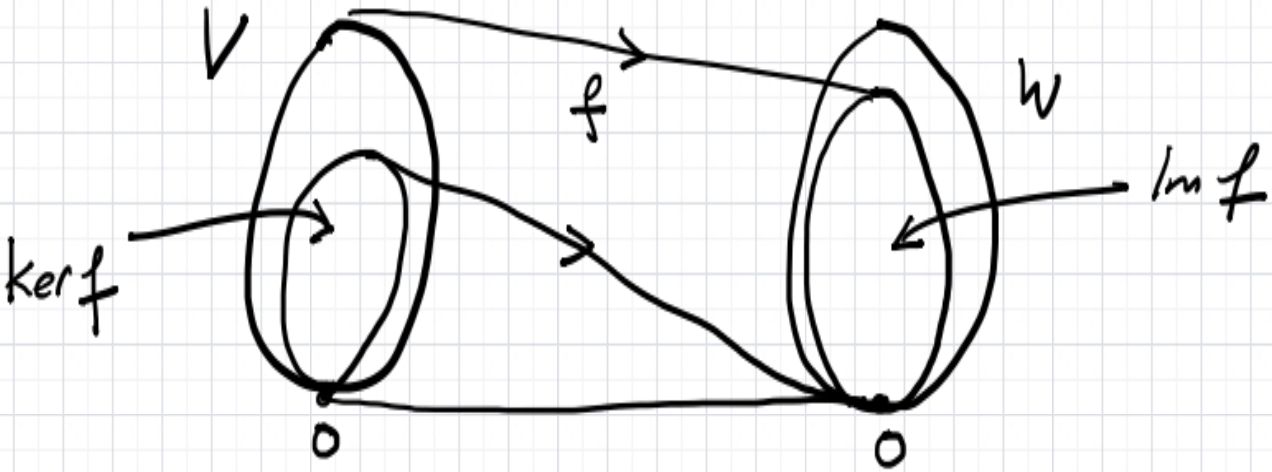
Lecture #17 Kernel & Image

Def$^{\underline{n}}$: Let $f: V \longrightarrow W$. Then

$$\ker f = \{v \mid f(v) = 0\} \subset V$$



$$\operatorname{Im} f = \{f(v) \mid v \in V\} \subset W$$

Theorem $\ker f$ and $\operatorname{Im} f$ are subspaces of V & W, respectively.

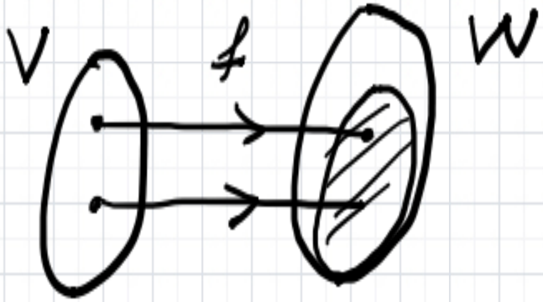
We proved this already

Remark $\ker f \equiv \operatorname{null} f$. $\operatorname{Im} f \equiv f(V) \equiv \operatorname{ran} f$.

Definition: $f : V \to W$ is called <u>injective</u> if

$$f(v) = f(u) \Rightarrow u = v$$

Remark: This is sometimes called $1:1$



f is "injecting" a copy ~~of~~ of V into W

Theorem Let $f : V \xrightarrow[\text{linear}]{} W$. Then $\ker f = \{0\} \Leftrightarrow f$ injective

Proof ($\Rightarrow$) Suppose $\ker f = \{0\}$ and $f(u) = f(v)$.

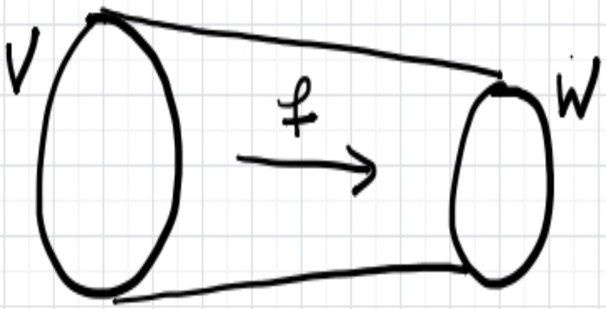
Then $0 = f(u) - f(v) \underset{\text{LINEAR}}{=} f(u-v) \Rightarrow u - v \in \ker f = \{0\} \Rightarrow u = v$ ✓

($\Leftarrow$) Suppose f is injective and $u \in \ker f$. Then $f(u) = 0$.

But $f(0) = f(0 \cdot 0) = 0 \cdot f(0) = 0 \Rightarrow f(u) = f(0) \Rightarrow u = 0$ QED

Ex $\frac{d}{dx} : \mathbb{F}[x] \to \mathbb{F}[x]$ is <u>not</u> $1:1$ b/c $p(x) = c \in \ker \frac{d}{dx}$.

Def$^{\underline{n}}$ We say $f: V \to W$ is surjective if $\operatorname{Im} f = W$.

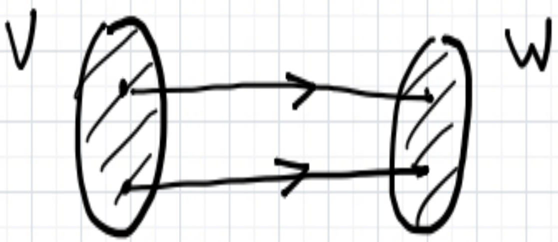


i.e. $\forall\, w \in W,\ \exists\, v \in V$ such that $f(v) = w$.

$*$ v need <u>not</u> be unique.

EX $\frac{d}{dx}: \mathbb{F}[x] \to \mathbb{F}[x]$ b/c $\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = \frac{d}{dx}\left(\alpha_0 x + \frac{\alpha_1}{2} x^2 + \cdots + \frac{\alpha_n}{n+1} x^n\right)$.

Def$^{\underline{n}}$ We say $f: V \to W$ is a bijection if f is 1:1 and onto.



Remark: When $f: V \xrightarrow[\text{Linear}]{} W$ (homomorphism) we call

$$\begin{cases} \text{surjective} - \text{epimorphism} \\ \text{injective} - \text{monomorphism} \\ \text{bijective} - \text{isomorphism} \end{cases}$$

Def$^{\underline{n}}$ If $f: V \to W$ is an isomorphism, we say V & W are <u>isomorphic</u>, $V \cong W$. In some sense they are the same vector space.

EX $\mathrm{Id}: V \to V$ is an isomorphism.

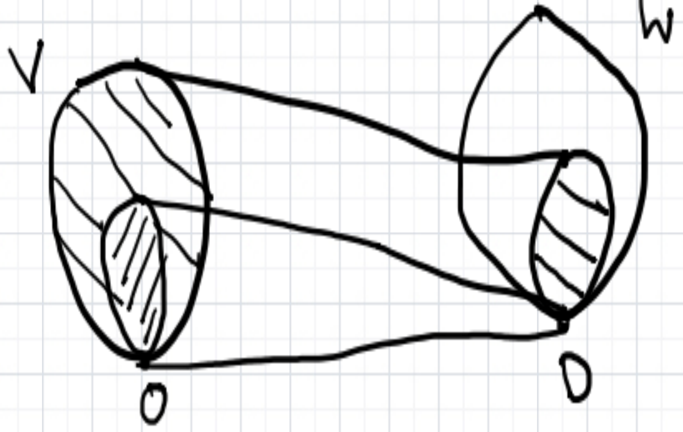
We call
$$\begin{tikzcd} V \arrow[loop above, "f"', "\text{linear}"] \end{tikzcd}$$
an endomorphism.

Bijective endomorphisms are called automorphisms.

The dimension formula

<u>Theorem</u> Let V be finite dimensional & $L : V \longrightarrow W$. Then

$$\dim \ker L + \dim \operatorname{Im} L = \dim V$$



<u>Proof:</u> $\ker L$ is a subspace of V so finite dimensional.

Let $v_1, \ldots, v_m$ be a basis for $\ker L$ where $m = \dim \ker L$.

Now use the basis extension theorem to produce a basis $v_1, \ldots, v_m, u_1, \ldots, u_n$ for V.

Here $m + n = \dim V$ so it remains to show $n = \dim \operatorname{im} L$.

It suffices to show $L(u_1), \ldots, L(u_n)$ is a basis for $\operatorname{im} L$.

Firstly for any $v \in V$ we have $v = \sum_{i=1}^{m} \alpha_i v_i + \sum_{i=1}^{n} \beta_i u_i$ so

$$L(v) = L\left(\sum \alpha_i v_i + \sum \beta_i u_i\right) = \sum \alpha_i \underbrace{\cancel{L(v_i)}}_{0} + \sum \beta_i L(u_i)$$

so $\operatorname{Im} L = \operatorname{span}(L(u_1), \ldots, L(u_n))$.

Now show $L(u_1), \ldots, L(u_n)$ are linearly independent.

suppose $\alpha_1 L(u_1) + \cdots + \alpha_n L(u_n) = 0$

Then $L(\alpha_1 u_1 + \dots + \alpha_n u_n) = 0$

$\Rightarrow \alpha_1 u_1 + \dots + \alpha_n u_n \in \ker L$

$\Rightarrow \alpha_1 u_1 + \dots + \alpha_n u_n$ is a linear contribution of $v_1, \dots, v_m$.

This is only possible if $\alpha_1 u_1 + \dots + \alpha_n u_n = 0$.

Thus $\alpha_1, \dots, \alpha_n$ must all vanish. $\square$