Lecture #18 Invertibility

Defⁿ $f : V \longrightarrow W$ is called <u>invertible</u> if there exists a map

$$g : W \longrightarrow V$$

such that

$$g \circ f = Id : \begin{array}{ccc} V & \longrightarrow & V \\ \in & & \in \\ v & \longmapsto & v \end{array}$$

and

$$f \circ g = Id : \begin{array}{ccc} W & \longrightarrow & W \\ \in & & \in \\ w & \longmapsto & w \end{array}$$

Remark Often call $g = f^{-1}$

Notice That if f is invertible, $f(u) = f(v) \underset{\text{act with } f^{-1}}{\Rightarrow} u = v$ 1:1

Moreover $W \ni w = f^{-1}(f(w)) \Rightarrow$ onto

Thus invertible maps are bijections. The converse holds ~~too~~:

Let $w \in W$, then b/c f is onto $\exists\, v \in V$ s.t. $f(v) = w$.

Because f is 1:1 v is unique. Call this $v = f^{-1}(w)$.

Clearly $f(f^{-1}(w)) = w$ by construction. Moreover for $f^{-1}(f(v))$ we have $w = f(v)$ in the above construction, so must search for the unique v' s.t. $f(v') = w = f(v)$.

This is $v' = v$ QED

EXERCISE?

Theorem f bijection $\Longleftrightarrow$ f invertible.

Theorem Let $f : V \xrightarrow[\text{linear}]{} W$ be invertible. Then $f^{-1} : W \xrightarrow[\text{linear}]{} V$.

Proof
$$
\begin{aligned}
f^{-1}(\alpha w + \beta w') &= f^{-1}(\alpha f(v) + \beta f(v')) \quad \left(\text{for unique } v, v' \text{ subject to } v = f^{-1}(w),\ v' = f^{-1}(w')\right) \\
&= f^{-1}(f(\alpha v + \beta v')) \\
&= Id(\alpha v + \beta v') \\
&= \alpha v + \beta v' \\
&= \alpha f^{-1}(w) + \beta f^{-1}(w').
\end{aligned}
$$

Theorem $f : V \xrightarrow{\text{linear}} W$ invertible $\Rightarrow \dim V = \dim W$

Proof The dimension formula says

$$\dim V = \underbrace{\dim \ker V}_{\text{inj.}} \underset{0}{\overset{\shortparallel}{}} + \dim \operatorname{Im} V \underset{W}{\overset{\shortparallel}{}}$$

$$\dim \ker V = 0, \quad \dim \operatorname{Im} V = W$$

QED

Theorem: $f : V \xrightarrow[\text{(isom)}]{\cong} W \iff f : V \xrightarrow[\substack{\text{linear} \\ \text{(monom.)}}]{1:1} W \ \& \ \dim V = \dim W$

$\iff f : V \xrightarrow[\substack{\text{linear} \\ \text{(epim.)}}]{\text{onto}} W \ \& \ \dim V = \dim W$

EXERCISE Proof

Theorem: $\dim V = \dim W \iff V \cong W$

Proof ($\Leftarrow$) was done above

($\Rightarrow$) Here we need to construct an isomorphism.

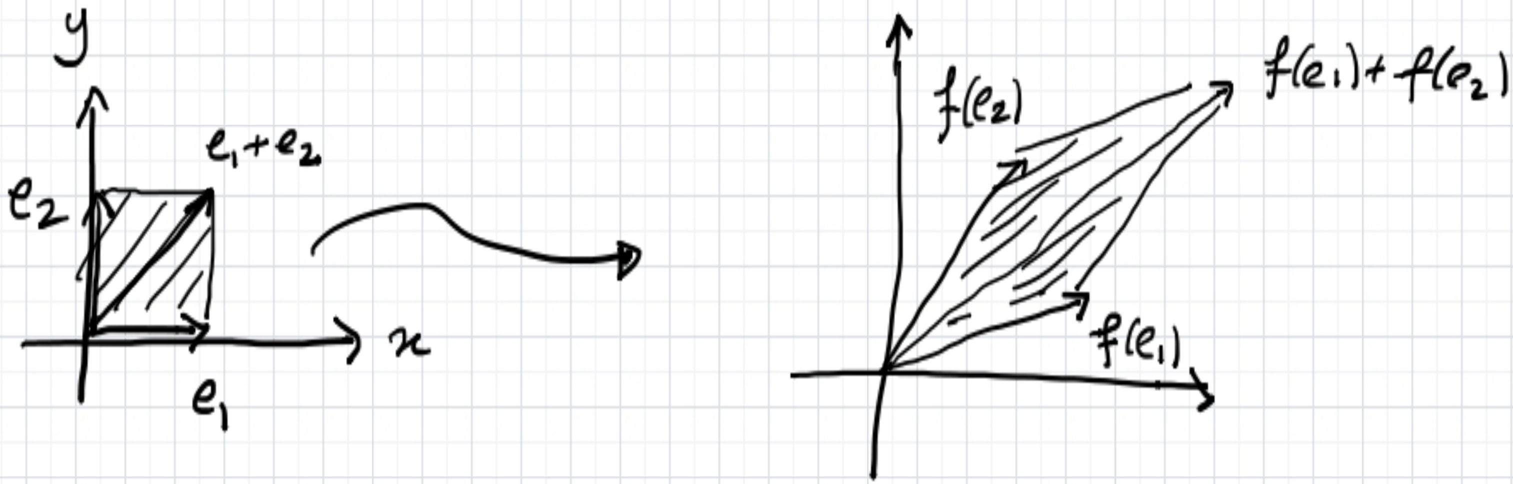
Pick bases $v_1, \dots, v_m$ & $w_1, \dots, w_m$ for V & W.

Set $f(v_i) = w_i$. Now show this map is onto and 1:1. The 1^{st} uses that $w_1, \dots, w_m$ spans W, the 2nd relies on linear independence of v_i. QED

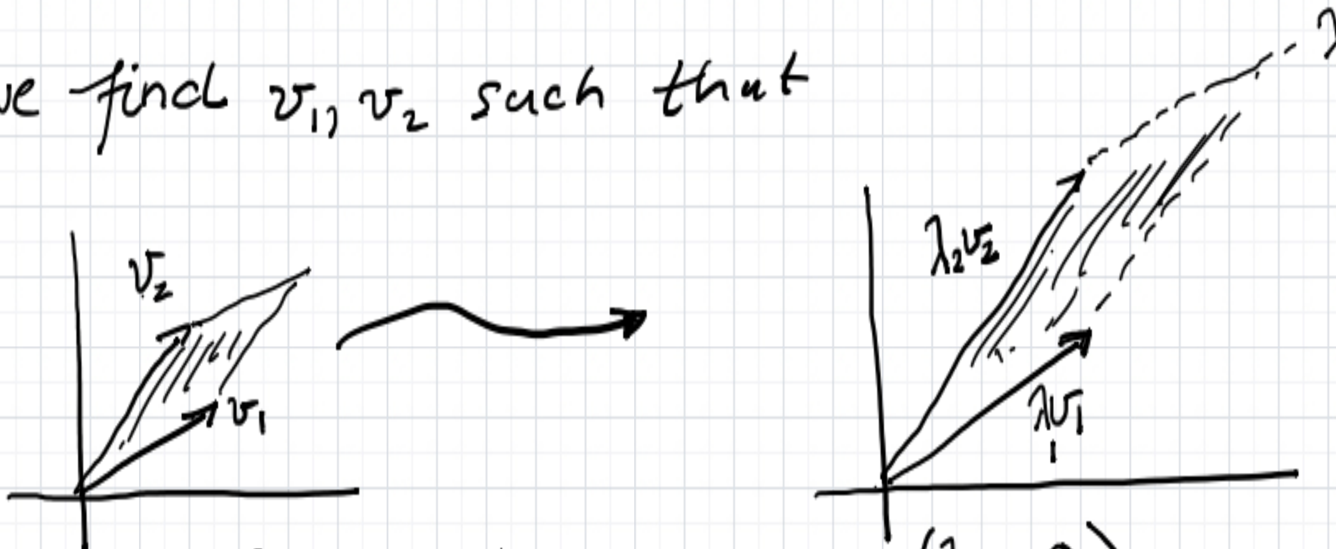
EXERCISE Fill in details.

Eigenvalues & Eigenvectors (ch. 7)

Consider $f: \mathbb{R}^2 \xrightarrow[\text{linear}]{} \mathbb{R}^2$



Can we find v_1, v_2 such that



BETTER DRAWN ON ONE GRAPH IN COLOR

In this basis $\begin{pmatrix} f(v_1) & f(v_2) \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

More over $\vec{x} \in \mathbb{R}^2$ is $\vec{x} = (e_1 \;\; e_2)\begin{pmatrix} x \\ y \end{pmatrix}$ and $f(\vec{x}) = (e_1 \;\; e_2)\, M(f) \begin{pmatrix} x \\ y \end{pmatrix}$

general matrix

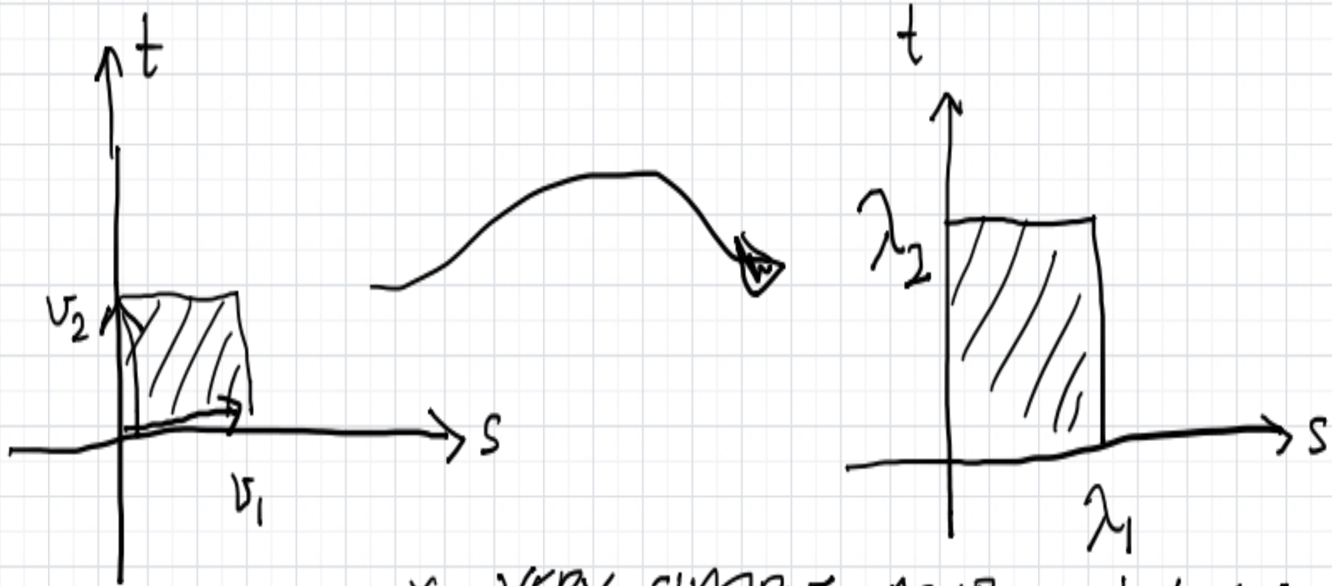
$= xe_1 + ye_2 \leftarrow M(\vec{x})$

but can also write $\vec{x} = (v_1 \;\; v_2)\begin{pmatrix} s \\ t \end{pmatrix}$ and $f(\vec{x}) = (v_1 \;\; v_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}$

$= sv_1 + tv_2$

$= (v_1 \;\; v_2)\begin{pmatrix} \lambda_1 s \\ \lambda_2 t \end{pmatrix}$

$= \lambda_1 s v_1 + \lambda_2 t v_2$



* VERY SIMPLE MAP — INVARIANT CHARACTERIZATION OF f

WANT TO SOLVE $f(v) = \lambda v$ OR $M(f)\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ //

The above pictures are examples of 1 dimensional <u>invariant subspaces</u>:

$$\text{If} \quad f: V \xrightarrow{\text{linear}} V \quad \text{and} \quad f(v) = \lambda v \qquad \lambda \neq 0$$

$$\text{then } \operatorname{span}(f(v)) = \operatorname{span}(v) \quad .$$

<u>Def</u>: Let $f: V \xrightarrow[\text{linear}]{} V$. We call U an <u>invariant subspace</u> if $f(u) \in U \quad \forall \; u \in U$.

<u>Remark</u> This is a <u>set statement</u>. $f(U) \subset U$.

<u>EX</u> $\ker f$. $\quad f(\ker f) = \{f(v) \mid v \in \ker f\} = \{0 \mid v \in \ker f\} = \{0\}$
$\subset \ker f$

$f(V) \qquad f(f(V)) = \{f(f(v)) \mid v \in V\} \subset \{f(v) \mid v \in V\} = f(V).$