Lecture # 19 Eigenvalues and eigenvectors

Def^{n} Let $L : V \xrightarrow[\text{linear}]{} V$. Then call $\lambda \in \mathbb{F}$ an <u>eigenvalue</u> of L if $\exists\, v \neq 0$ such that

$$\boxed{Lv = \lambda v}$$

The vector v is called the <u>eigenvector</u> associated to λ.

Remarks:

* If V has a basis of eigenvalues $M[L]$ is diagonal:

$$\big(L(v_1) \cdots L(v_n)\big) = (\lambda_1 v_1 \quad \lambda_2 v_2 \cdots \lambda_n v_n) = (v_1 \cdots v_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

* If v is an eigenvector with eigenvalue λ, then so is μv for any $0 \neq \mu \in \mathbb{F}$.

* Given a basis of eigenvectors L^{-1} and L^k $k \in \{2, 3, \ldots\}$ are immediately computable:

$$L^k(v_j) = (\lambda_j)^k v_j$$

$$L^{-1}(v_j) = \frac{1}{\lambda_j} v_j \quad \text{(if it exists)}$$

* If v is an eigenvector

$$\mathrm{span}(L(v)) \subset \mathrm{span}(v)$$

This is an example of an <u>invariant subspace</u>:

$U \subset V$ such that $L(U) \subset U$.

* $\mathrm{Id}: V \longrightarrow V$ has eigenvalue 1 and every

$$\begin{array}{ccc} V & \longrightarrow & V \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & v \end{array}$$

subspace is invariant.

<u>Def$^{\underline{n}}$</u> Let $L: V \xrightarrow[\text{linear}]{} V$ with eigenvalue λ. Then

$\ker(L - \lambda\, \mathrm{Id})$ is called an <u>eigenspace</u> of L.

<u>Remarks</u> * If $v \in \ker(L - \lambda\, \mathrm{Id})$ then

$$0 = (L - \lambda\, \mathrm{Id})v = Lv - \lambda v \iff Lv = \lambda v$$

so every element of an eigenspace is an associated eigenvector.

* λ is an eigenvalue $\iff$ $L - \lambda \mathrm{Id}$ is <u>not</u> injective.

$\iff$ $L - \lambda \mathrm{Id}$ is <u>not</u> surjective.

(when V is finite dimensional)

<u>Theorem</u> Let $L : V \xrightarrow[\text{linear}]{} V$ with <u>distinct</u> eigenvalues $\lambda_1, \ldots, \lambda_m$ and associated eigenvectors $v_1, \ldots, v_m$. Then $v_1, \ldots, v_m$ are linearly independent.

<u>Proof</u>: By contradiction assume (Linear dependence lemma)

$$v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} \qquad k \in \{2, \ldots, m\},\ v_1, \ldots, v_{k-1} \text{ linearly indep}^{\underline{t}}$$

act with

$$\underset{L}{\Rightarrow} \quad \lambda_k v_k = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_{k-1} \lambda_{k-1} v_{k-1}$$

$$\Rightarrow \quad 0 = \alpha_1 (\underbrace{\lambda_1 - \lambda_k}_{\neq 0}) v_1 + \cdots + \alpha_{k-1} (\underbrace{\lambda_{k-1} - \lambda_k}_{\neq 0}) v_{k-1}$$

$\Rightarrow$ $\alpha_1, \ldots, \alpha_{k-1}$ all vanish ($v_1, \ldots, v_{k-1}$ indep$^{\underline{t}}$) $\Rightarrow v_k = 0$ $\lightning$ <u>QED</u>

Corollary $L : V \xrightarrow[\text{linear}]{} V$ has at most n distinct eigenvalues.

Remark If L has $\dim V$ distinct eigenvalues, it has a basis of eigenvectors.

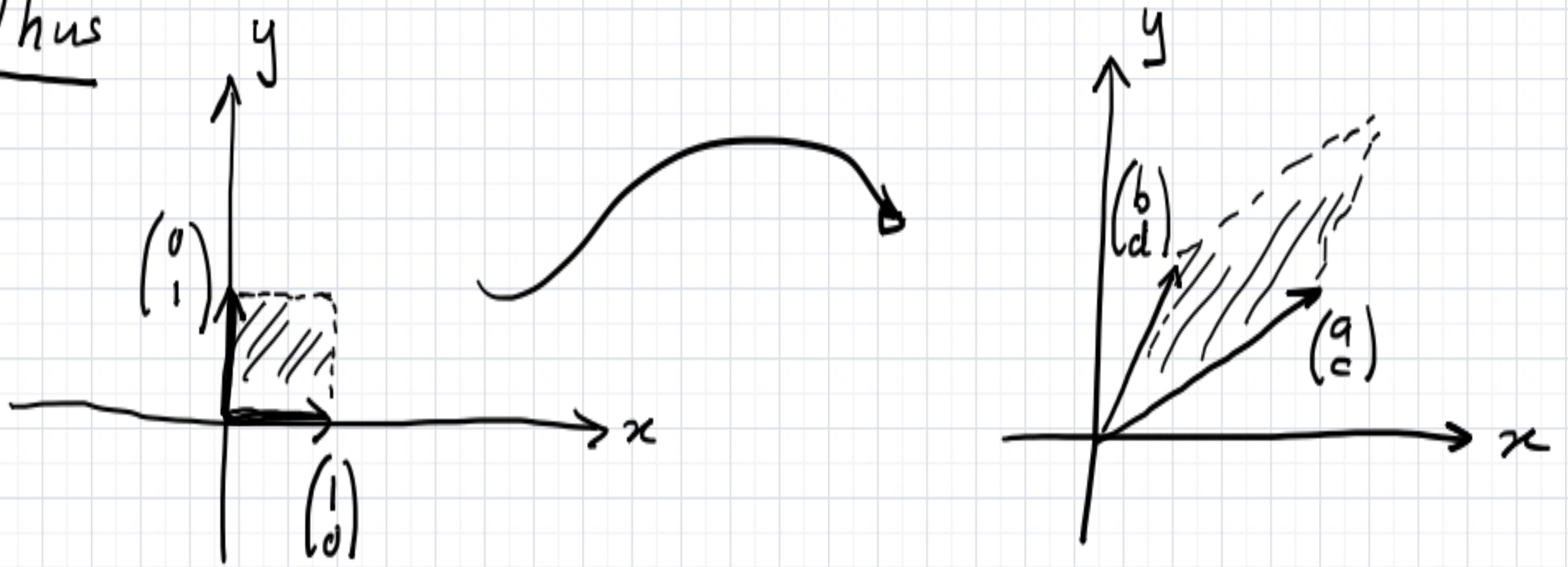
Diagonalizing 2x2 matrices

Let $\dim V = 2$ and $L : V \longrightarrow V$

Then $V \cong \mathbb{R}^2$. Suppose $M[L] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in a basis f_1, f_2. Then

$$\vec{x} = (f_1 \ f_2)\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto (f_1 \ f_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (f_1 \ f_2)\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Thus



$\Rightarrow$ $\mathrm{span}(L(f_1), L(f_2)) = V$ whenever the area of the above parallelogram is non-zero $\Leftrightarrow$ L invertible. But this is

$$|ad - bc|.$$ (Easily checked $\bar{c}$ a little trig!)

Define
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc$$

For L to have an eigenvector must solve
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Or better $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

This requires

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\Leftrightarrow \quad (a-\lambda)(d-\lambda) - bc = 0$$

$$\Leftrightarrow \quad \lambda^2 - \underbrace{(a+d)}_{:\mathrm{tr} M}\lambda + \underbrace{ad-bc}_{\det M} = 0$$

This has solutions

$$\lambda_\pm = \frac{\mathrm{tr} M \pm \sqrt{(\mathrm{tr} M)^2 - 4\det M}}{2}$$

Then solve the associated linear system $\begin{pmatrix} a-\lambda_\pm & b \\ c & d-\lambda_\pm \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$ for the eigenvectors.

<u>EX</u> Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ by a rotation of $\measuredangle\,\theta$ about the origin.
Find the matrix of R w.r.t. e_1, e_2 as well as its eigenvalues.