Lecture #20

* Review bit & exponential problem

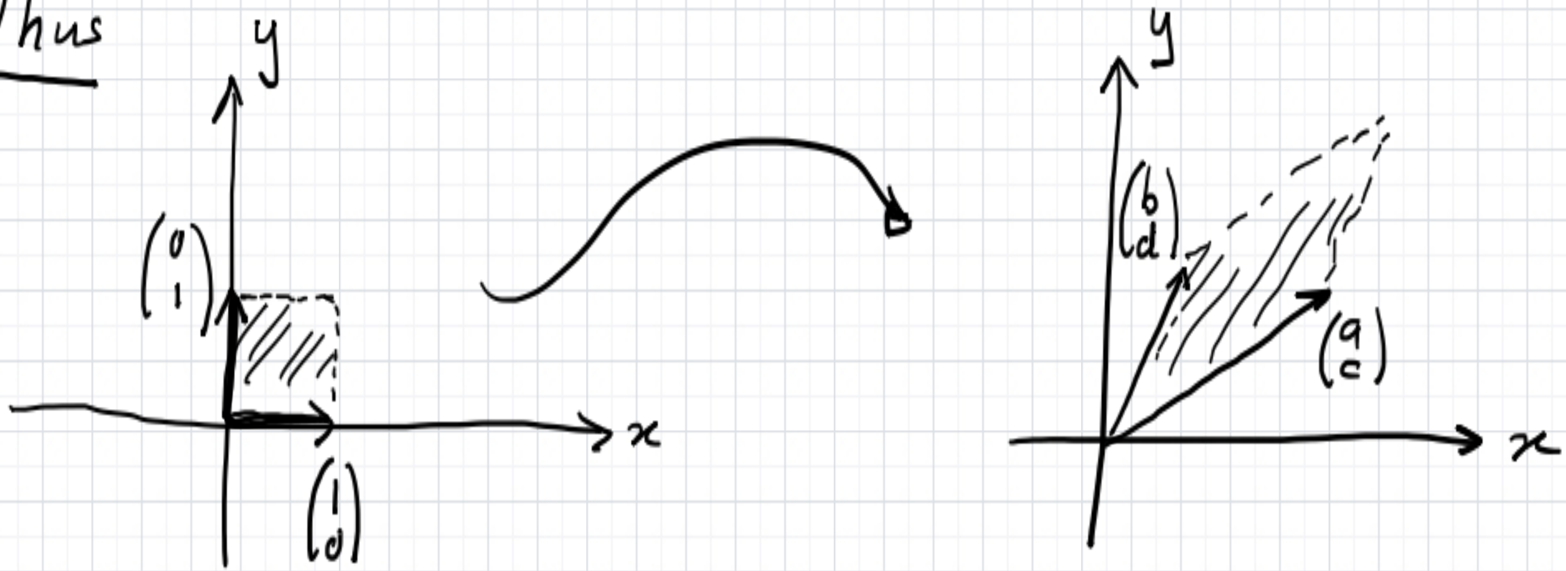
Diagonalizing 2x2 matrices

Let $\dim V = 2$ and $L : V \longrightarrow V$

Then $V \cong \mathbb{R}^2$. Suppose $M[L] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in a basis f_1, f_2. Then

$$v = (f_1 \ f_2)\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto (f_1 \ f_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (f_1 \ f_2)\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Thus



$\Rightarrow \operatorname{span}(L(f_1), L(f_2)) = V$ whenever the area of the above parallelogram is non-zero $\Leftrightarrow L$ invertible. But this is

$$|ad - bc|.$$

(Easily checked $\bar{c}$ a little trig!)

Define $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc$

For L to have an eigenvector must solve

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Or better $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

This requires

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\Leftrightarrow \quad (a-\lambda)(d-\lambda) - bc = 0$$

$$\Leftrightarrow \quad \lambda^2 - \underbrace{(a+d)}_{:\mathrm{tr} M}\lambda + \underbrace{ad-bc}_{\det M} = 0$$

This has solutions

$$\lambda_\pm = \frac{\mathrm{tr} M \pm \sqrt{(\mathrm{tr} M)^2 - 4\det M}}{2}$$

Then solve the associated linear system $\begin{pmatrix} a-\lambda_\pm & b \\ c & d-\lambda_\pm \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$ for the eigenvectors.

<u>EX</u> Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ by a rotation of $\measuredangle\, \theta$ about the origin.
Find the matrix of R w.r.t. e_1, e_2 as well as its eigenvalues.

Determinants

EX Let $M[L] = \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix}$ where $L: V \xrightarrow[\text{linear}]{} V$, $\dim V = 2$.

We saw L invertible $\iff \det M = m^1_1 m^2_2 - m^1_2 m^2_1 \neq 0$

In fact, for $\dim V = 3$, a similar criterion exists

$$\det M = m^1_1 m^2_2 m^3_3 - m^1_2 m^2_1 m^3_3 + m^1_2 m^2_3 m^3_1 - m^1_3 m^2_2 m^3_1 + m^1_3 m^2_1 m^3_2 - m^1_1 m^2_3 m^3_2$$

Notice this formula is obtained by permuting column numbers with signs.

Permutations

Set of bijections

$$\sigma : \{1, 2, .., n\} \longrightarrow \{1, 2, .., n\}$$

Common notations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix} \quad \text{OR JUST} \quad \sigma(1)\ \sigma(2)\ \cdots\ \sigma(n)$$

The set of all σ is called S_n, it has $n!$ elements and forms a non-abelian group with multiplication defined by composition. Groups will be the main focus in 150A.

The sign function

All we need to know about S_n is that there is a well-defined <u>group homomorphism</u>

$$\mathrm{sign} : S_n \longrightarrow \{+1, -1\}$$

by counting the number of pairwise swaps required to reach the identity permutation

$$\mathrm{sgn}(\sigma) = \begin{cases} +1 & \text{even \# swaps} \\ -1 & \text{odd \# swaps} \end{cases}$$

<u>EX</u> $\mathrm{sgn}(\underset{-}{65}\ 14\ \underset{-}{32}) = +1.$

The determinant Let $M = (m^i_j)$ be an $n \times n$ matrix, then

$$\boxed{\det M = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \cdot m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}}$$

* Computation involves $n!$ terms, half +ve, half -ve $\rightsquigarrow$ <u>very</u> inefficient.

* Let $I = M[\mathrm{Id}] = (\delta^i_j)$ $\quad \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\det I = 1$$

* Study $\det M'$ where M equals M' save that a pair of rows are swapped!