Lecture#20 Determinants (Quiz theme – e'values & e'vectors)

EX Let $M[L] = \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix}$ where $L : V \xrightarrow[\text{linear}]{} V$, $\dim V = 2$.

We saw L invertible $\Longleftrightarrow$ $\det M = m^1_1 m^2_2 - m^1_2 m^2_1 \neq 0$

In fact, for $\dim V = 3$, a similar criterion exists

$$\det M = m^1_1 m^2_2 m^3_3 - m^1_2 m^2_1 m^3_3 + m^1_2 m^2_3 m^3_1 - m^1_3 m^2_2 m^3_1 + m^1_3 m^2_1 m^3_2 - m^1_1 m^2_3 m^3_2$$

Notice this formula is obtained by permuting column numbers with signs.

Permutations

Set of bijections

$$\sigma : \{1, 2, .., n\} \longrightarrow \{1, 2, .., n\}$$

Common notations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) \cdots & \sigma(n) \end{pmatrix} \quad \text{OR JUST} \quad \sigma(1)\ \sigma(2)\ \cdots\ \sigma(n)$$

The set of all σ is called S_n, it has $n!$ elements and forms a non-abelian group with multiplication defined by composition. Groups will be the main focus in 150A.

The sign function

All we need to know about S_n is that there is a well-defined group homomorphism

$$\mathrm{sign} : S_n \longrightarrow \{+1, -1\}$$

by counting the number of pairwise swaps required to reach the identity permutation

$$\mathrm{sgn}(\sigma) = \begin{cases} +1 & \text{even \# swaps} \\ -1 & \text{odd \# swaps} \end{cases}$$

EX $\mathrm{sgn}(6\,5\,1\,4\,3\,2) = +1.$

The determinant Let $M = (m^i_j)$ be an $n \times n$ matrix, then

$$\boxed{\det M = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \cdot m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}}$$

* Computation involves $n!$ terms, half +ve, half -ve $\rightsquigarrow$ very inefficient:

$\Rightarrow$ FIND CLEVER RULES FOR DETERMINANTS

* Let $I = M[\mathrm{Id}] = (\delta^i_j)$ $\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\det I = 1$$

* If M is triangular

$$M = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \underline{\text{OR}} \quad M = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_n \end{pmatrix}$$

$$\det M = \lambda_1 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i$$

<u>Proof</u> $\det M = m^1_1 m^2_2 \cdots m^n_n \underbrace{\pm \text{permutations}}_{\text{vanish}} = \lambda_1 \cdots \lambda_n$

* If $M' = M$ with a pair of rows swapped

$$\boxed{\det M' = -\det M}$$

Proof $\det M' = \sum_\sigma \operatorname{sgn}\sigma \cdot m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^j_{\sigma(i)} \cdots m^i_{\sigma(j)} \cdots m^n_{\sigma(n)}$

$j > i$

$= \sum_\sigma (-\operatorname{sgn}\tilde\sigma) \cdot m^1_{\tilde\sigma(1)} m^2_{\tilde\sigma(2)} \cdots m^i_{\tilde\sigma(i)} \cdots m^j_{\tilde\sigma(j)} \cdots m^n_{\tilde\sigma(n)}$

$\tilde\sigma$ is σ with $i \leftrightarrow j$

But $\sum_\sigma f(\sigma) = \sum_\sigma f(\tilde\sigma)$ so we are done.

$\boxed{\text{EXERCISE}}$ Define $M^T = (\bar m^i_j)$, the <u>transpose matrix</u> where

$$\bar m^i_j = m^j_i.$$

Use the above method to prove

$$\det M^T = \det M$$

* Corollaries If two rows or columns of M are the same $\det M = 0$

* Let M' equal M save for multiplying one row by μ.

Then

$$\det M' = \mu \det M$$

Proof This is more or less obvious from the definition because $m^i_j \to \mu m^i_j$ for some row number i.

* Let M' equal M with row i replaced by row i plus λ (row k). Then $\det M' = \det M$

because $m^i_j \to m^i_j + \lambda m^k_j$

det M

det of matrix $\bar{c}$ two equal rows.

We have established the effect of row operations on the determinant:

$$\begin{cases} M \xmapsto{R_i \leftrightarrow R_j} M' & \det M' = -\det M \\ M \xmapsto{R_i \to \lambda R_i} M' & \det M' = \lambda \det M \\ M \xmapsto{R_i \to R_i + \lambda R_k} M' & \det M' = \det M \end{cases}$$

Applying these to the identity gives "elementary matrices"

$$I \xmapsto{R_i \leftrightarrow R_j} E^i_j = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \qquad \det E^i_j = -1$$

$$I \xmapsto{R_i \to \lambda R_i} R^i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \det R^i(\lambda) = \lambda$$

$$I \xmapsto{R_i \to R_i + \mu R_k} S^i_k(\mu) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \ddots & \mu \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \qquad \det S^i_k(\mu) = 1$$

Claim $E^i_j M = M'$, $R^i(\lambda) M = M'$, $S^i_k(\mu) M = M'$ where
$$\begin{cases} M \xrightarrow{R_i \leftrightarrow R_j} M' \\ M \xrightarrow{R_i \to \lambda R_i} M' \\ M \xrightarrow{R_i \to R_i + \mu R_k} M' \end{cases}$$

I.e. left multiplication by elementary matrices performs row operations — this is clocked by matrix multiplication.

Moreover if E is any of E^i_j, $R^i(\lambda)$, $S^i_k(\mu)$ we see

$$\boxed{\det EM = \det E \det M}$$

But Gauss–Jordan elimination performs row operations to bring M to the identity or RREF. For the former case this means $M = E_1 \cdots E_k I$ where $E_1, \ldots, E_k$ are a sequence of elementary matrices. Thus

$$\det M = \det E_1 \cdots \det E_k$$

Similarly, if

$$N = F_1 \cdots F_\ell$$

for some other matrix N. Thus

$$\det MN = \det E_1 \cdots E_k F_1 \cdots F_\ell = \det E_1 \cdots \det E_k \det F_1 \cdots \det F_\ell$$
$$= \det M \det N .$$

When M is singular, $RREF(M)$ has a row of zeros so $\det M = 0$, but the above conclusion still holds.

Summarise the above discussion by

Theorem Let M, N be the matrices of linear transformations

$$L : V \to V, \quad K : V \to V.$$

Then i) L non-invertible $\iff \det M = 0$

ii) $$\det MN = \det M \det N$$