Lecture#22 Determinants ctd.

Recall, the eigenvalue problem $Lv = \lambda v$ required a characterization of when

$L - \lambda Id$ is invertible, achieved via $\det M[L - \lambda Id]$.

$$\det M = \sum_{\sigma} \operatorname{sgn}\sigma \, m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}$$

Row operations required for Gauß-Jordan elimination:

Row swap $\quad R_i \leftrightarrow R_j$

Scalar Multiply $\quad R_i \rightarrow \lambda R_i$

Addition $\quad R_i \rightarrow R_i + \lambda R_j$

Row operation	Elementary Matrix E = Row op. on I	Change in determinant	$\det E$
$R_i \leftrightarrow R_j$	$E^i_j = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$	$\det M' = -\det M$	-1
$R_i \to \lambda R_i$	$R^i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$	$\det M' = \lambda \det M$	λ
$R_i \to R_i + \lambda R_j$	$S^i_j(\lambda) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & \cdots & \lambda & \\ & & & \ddots & \vdots & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$	$\det M' = \det M$	1

In all cases $M' = EM$ & $\det M' = \det E \cdot \det M$

Two cases ① M invertible

sequence of row op.s

Gauß- Jordan elimination $M \sim \dots \sim I$

$$\Rightarrow \quad M = E_1 E_2 .. E_k I$$ sequence of elementary matrices

$$\Rightarrow \quad \det M = \det E_1 \det E_2 \dots \det E_k$$

② M not invertible

$$\Rightarrow \quad M = E_1 E_2 \dots E_k \, RREF(M)$$

where $RREF(M)$ has a row of zeros.

$$\Rightarrow \boxed{\det M = 0 \iff M \text{ not invertible.}}$$

Now suppose M & N are both invertible, then

$$\left.\begin{aligned} M &= E_1 \cdots E_k \\ N &= F_1 \cdots F_l \end{aligned}\right\} \text{ sequences of elementary matrices}$$

$$\Rightarrow \quad \det M = \det E_1 \cdots \det E_k$$

$$\det N = \det F_1 \cdots \det F_l$$

But $\det MN = \det E_1 \cdots E_k F_1 \cdots F_l$

$$= \det E_1 \cdots \det E_k \det F_1 \cdots \det F_l$$

$$\Rightarrow \boxed{\det MN = \det M \det N}$$

When either M or N is singular, easy to check the same holds.

Corollary: If M^{-1} exists, $\det M^{-1} = \frac{1}{\det M}$

Application: The characteristic polynomial

If λ is an eigenvalue, $M - \lambda I$ is singular,

thus

$$\det(M - \lambda I) = 0$$

This quantity is called the characteristic polynomial

$$P(\lambda) = \det(M - \lambda I).$$

It is order n, thus over $\mathbb{C}$

$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are eigenvalues.

If a root λ_i repeats k times we say λ_i has <u>multiplicity k</u>.

Example:

Scientists at CERN notice that if their beam deviates from its target by an amount $(\delta x, \delta y)$, the deviation after the particles perform another loop is

$$(\delta x + 2\delta y,\ 3\delta x + 6\delta y).$$

For a fixed initial deviation, what happens to the beam after N loops?