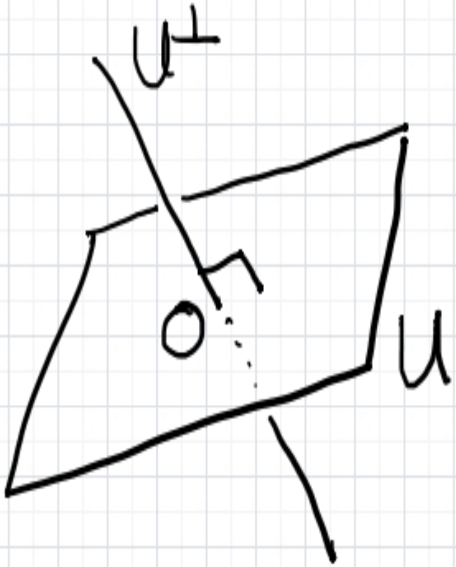
Lecture 26 $\rightsquigarrow$ change of basis worksheet.

Orthogonal Projections

V a vectorspace over $\mathbb{F}$.



Defn $U^\perp = \{w \in V \mid w \perp v \ \ \forall v \in V\}$

* Recall $\langle w, v \rangle = 0 \iff w \perp v$
* Easy to check $U^\perp$ is a subspace via the subspace theorem because

$$\langle \alpha w + \beta z, v \rangle = \alpha \langle w, v \rangle + \beta \langle z, w \rangle$$

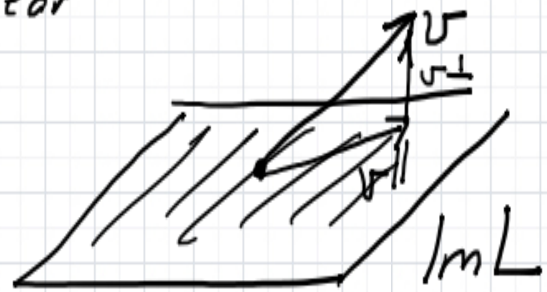
Theorem $U \oplus U^\perp = V$ & $(U^\perp)^\perp = U$.

Neither proof is difficult...

Least squares

Suppose $L : U \xrightarrow[\text{linear}]{} V$ and want to solve

$$L(u) = v$$



If $v \notin \operatorname{Im} L$ there is no solution. However, given an inner product for V we can uniquely split

$$v = v^\parallel + v^\perp$$

and $v^\parallel$ will be the best sol^n as measured by the inner product. I.e. minimize $\|v^\perp\|$, the error. Proven in 9.6.

Notice that we want to set $v^\parallel = L(u)$ thus require u obeys

$$\langle \underbrace{L(u')}_{\in \operatorname{Im} L}, \underbrace{v - L(u)}_{v^\perp} \rangle = 0 \quad \forall u' \in U$$

The above amounts to solving

$$L^{\dagger}(L(u) - v) = 0$$

or, if M is the matrix of L is some basis & we work over $\mathbb{R}$

$$\boxed{M^T M u = M^T v}$$

EXERCISE, SHOW THIS!

To define symbols, note that given $\langle , \rangle$ on V and $v \in V$ $\langle L(\cdot), v \rangle \in \mathcal{L}(U, \mathbb{F}) = U^*$ dual space.

But given $\alpha \in U^*$ we can uniquely define $u \in U$ via

$$\alpha(u') = \langle u, u' \rangle \quad \forall \; u' \in U$$

i.e. $\alpha(\cdot) = \langle u, \cdot \rangle$. Establishing this shows that $V \cong V^*$ canonically given $\langle \cdot, \cdot \rangle$.

The Spectral Theorem Ch 9.

Self-adjoint operators

$(V, \langle , \rangle)$ an inner product space, $\dim V < \infty$

Def$^{\underline{n}}$ Let $L : V \xrightarrow[\text{linear}]{} V$. Call the hermitean conjugate

$L^* : V \xrightarrow[\text{linear}]{} V$ the map defined by

$$\langle Lv, w \rangle = \langle v, L^* w \rangle$$

$\forall\ v, w \in V$.

Remark Also call L^* the adjoint of L. The above is well-defined because knowing $\langle Lv, w_i \rangle$ for (w_i) an ortho-normal basis completely determines

$$Lv = \sum_i \langle Lv, w_i \rangle w_i \quad \text{using } \langle w_i, w_j \rangle = \delta_{ij}$$

Def$^{\underline{n}}$ If $L = L^*$ we say L is __hermitean__ or __self-adjoint__ (for infinite dimensional case these two notions can differ).

__Remark__ In quantum mechanics, self-adjoint operators correspond precisely to measurable quantities, their eigenvalues being the possible values the measurement could produce.

__Theorem__ The eigenvalues of a self adjoint linear operator are real.

__Remark__ Desirable property for observable quantities.

__Proof__ Suppose $Lv = \lambda v$. Then

$$\langle Lv, v\rangle = \lambda \langle v, v\rangle = \lambda \|v\|^2$$

$$\Rightarrow \quad \bar{\lambda} \|v\|^2 = \overline{\langle Lv, v\rangle} = \langle v, Lv\rangle$$

$$= \langle L^* v, v\rangle = \langle Lv, v\rangle = \lambda \|v\|^2$$

Since $\|v\|^2 \neq 0$ $\lambda = \bar{\lambda}$.

__Def$^{\underline{n}}$__ Normal operator $L : V \xrightarrow[\text{linear}]{} V$

$$[L, L^*] := LL^* - L^*L = 0$$

__Next time__ This is the iff condition for a basis of eigenvectors.

__Remark__ Self adjoint operators are __always__ normal!