Lecture 27 The exam

Guide: Chapters 1, 2, 4–9 comprehensive

- definitions
- theorems
- homeworks

Chapters 10, 11 read over these (guided question only)

Q1 Bases, lin. indep., spanning, morphisms

Q2 Determinants

Q3 Vector Spaces

Q4 Inner Product Spaces

Q5 Inner Product Spaces, kernel, image, dimension formula

Q6 Spectral Theorem, Eigenvalues & Eigenvectors

LEAST SQUARES

Data

v	t
11	1
19	2
31	3

Theoretical Model

$$v = at + b$$

This is a linear systems problem

$$L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad L(x) = v$$

because unknowns span $\mathbb{R}^2$ & we have 3 data points

$$\begin{aligned} 11 &= a \cdot 1 + b \\ 19 &= a \cdot 2 + b \\ 31 &= a \cdot 3 + b \end{aligned} \implies \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix}$$

<u>Thus</u> $M = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ is the matrix of a

linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^3$

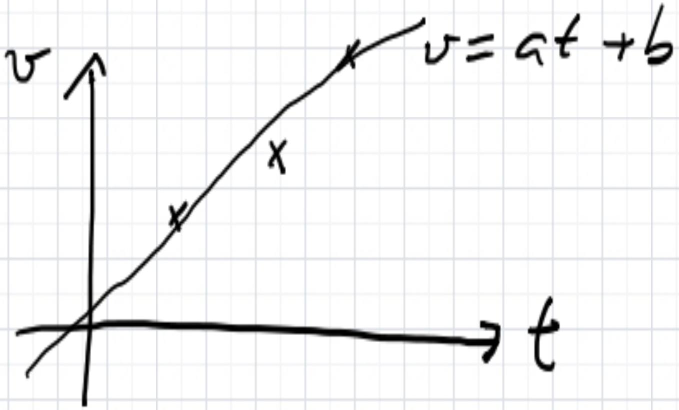
and $V = \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix}$ is the components of a

vector in $\mathbb{R}^3$, $X = \begin{pmatrix} a \\ b \end{pmatrix}$ —— // —— in $\mathbb{R}^2$.

We can check that $MX = V$ has <u>NO</u> solution

$$\left(\begin{array}{cc|c} 1 & 1 & 11 \\ 2 & 1 & 19 \\ 3 & 1 & 31 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 11 \\ 0 & -1 & -3 \\ 0 & -2 & -2 \end{array}\right) \left.\begin{matrix} b = 3 \\ b = 1 \end{matrix}\right\} \lightning$$

Now need to decide how to measure if a solution is optimal



Need an inner product on the space of velocity measurements

$$(v_1, v_2, v_3) = v$$

This is a <u>choice</u>. For example, if you believe the first measurement is very accurated, and the others are shaky you might use

$$\|v\|^2 = 10 v_1^2 + v_2^2 + v_3^2$$

This would give a preference to best fits close to the first point.

It is also useful to have an inner product, not just a norm. Thus, set

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1)$$

and use

$$\langle e_i, e_j \rangle = \delta_{ij} = e_i \cdot e_j \text{ (standard dot product).}$$

<u>Notice</u> we only need an inner product on the data space, i.e., <u>codomain</u>.

<u>Now</u> $\quad L(x) = v \quad$ had no solution

But last time suggested least squares equation

$$L^+(L(x)) = L^+(v)$$

Last time defined L^+ using dual space. A simpler way is as follows. The columns of M span $L(\mathbb{R}^2)$ in the e_1, e_2, e_3 basis.

Consider $M^T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

Then the equation

$$M^T Y = 0 \quad \text{says} \quad \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

so dot products b/w Y & columns of M vanish $\Rightarrow Y \in (L(\mathbb{R}^2))^\perp$

The matrix M^T is the matrix of L^+.

Thus we have to solve

$$M^T M X = M^T V$$

Notice that if $\ker M = 0$, $M^T M$ is invertible b/c

$$M^T M X = 0 \Rightarrow X^T M^T M X = (MX)^T (MX) = (MX)\cdot(MX)$$

$$= \|MX\|^2 \Rightarrow MX = 0 \Rightarrow X = 0$$

Thus $X = (M^T M)^{-1} M^T V$

$$= \left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix}$$

Conundrum was pushed

$$= \frac{1}{6} \begin{pmatrix} -3 & 0 & 3 \\ 8 & 2 & -4 \end{pmatrix} \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 60 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ \frac{1}{3} \end{pmatrix}$$

$$\boxed{v = 10t + \frac{1}{3}}$$