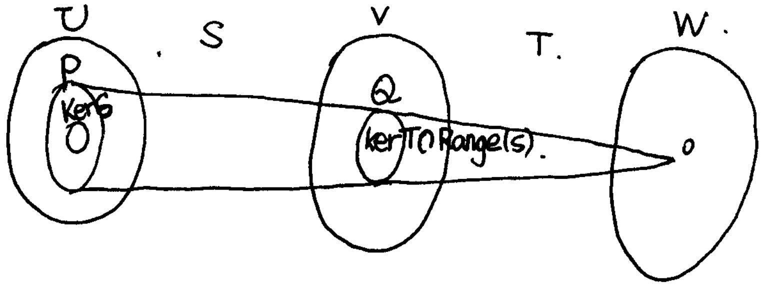
PWE 7.



Let $Q = \ker(T) \cap \text{Range}(S) \subseteq V$

Q is a subspace of V because Q is the intersection of two subspaces.

Let $P = \{u \in U \mid S(u) \in Q\} = S^{-1}(Q) \subseteq U$.

Next, we prove P is a subspace of U.

① Take $u_1, u_2 \in P$. $S(u_1) \in \ker T \cap \text{Range}(S)$ $S(u_2) \in \ker T \cap \text{Range}(S)$

which implies $T \circ S(u_1) = 0$. $T \circ S(u_2) = 0$.

Then $u_1 + u_2$, $T \circ S(u_1+u_2) = T \circ (S(u_1) + S(u_2)) = T \circ S(u_1) + T \circ S(u_2) = 0$

which implies $S(u_1+u_2) \in \ker T$.

It's easy to see that $S(u_1+u_2) \in \text{Range}(S)$

Then $S(u_1+u_2) = \ker T \cap \text{Range}(S)$.

$\Rightarrow u_1 + u_2 \in P. = S^{-1}(\ker T \cap \text{Range}(S))$

② Take $u \in P$. $S(u) \in \ker T \cap \text{Range}(S) \Rightarrow T \circ S(u) = 0$

Then αu. $T \circ S(\alpha u) = T(\alpha S(u)) = \alpha T \circ S(u) = 0$.

It's easy to see that $S(\alpha u) \in \text{Range}(S)$

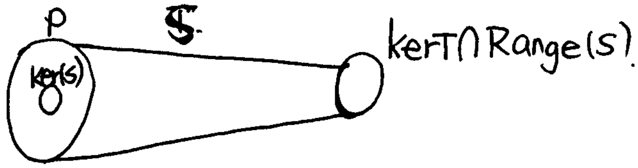
Then $S(\alpha u) \in \ker T \cap \text{Range}(S)$

$\Rightarrow \alpha u \in P = S^{-1}(\ker T \cap \text{Range}(S))$

Since $\ker(T \circ S)$ contains all vectors in V whose images are in $\ker T$.

$P = \ker(T \circ S)$.

Consider the restriction of S on P : $S|_P$.



First : $S|_P$ is surjective.

Second : $\ker(S) \subseteq P$ and $\ker(S|_P) = \ker(S)$.

By dimension formula.

$$\dim P = \underset{\dim \ker(S)}{\underset{\parallel}{\dim \ker(S|_P)}} + \underset{\dim(\ker T \cap \operatorname{Range}(S))}{\underset{\parallel}{\dim \operatorname{Range}(S|_P)}}$$

Since $\dim(\ker(T) \cap \operatorname{Range}(S)) \leq \dim(\ker T)$

$$\dim P = \dim(\ker(T \circ S)) \leq \dim \ker(S) + \dim(\ker(T)). \quad \square$$