# CHANGE OF BASIS WORKSHEET

Let $x \in V$ & $L : V \xrightarrow[\text{linear}]{} W$ & $(v_1, \dots, v_n)$, $(w_1, \dots, w_m)$ be bases for $V$ & $W$, respectively.

Suppose $x = (v_1 \; v_2 \; \cdots \; v_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x^n \end{pmatrix}$ & $L(x) = (w_1 \; w_2 \cdots w_m) M \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x^n \end{pmatrix}$

1. What size is the matrix $M$?

2. If $(v_1', v_2', \dots, v_n')$, $(w_1', w_2', \dots, w_m')$ are new bases for $V$ & $W$

   and

   $(v_1' \; v_2' \; \dots \; v_n') = (v_1 \; v_2 \; \cdots \; v_n) P$ & $(w_1' \; w_2' \dots w_m') = (w_1 \; w_2 \cdots w_m) Q$,

   what size are the matrices $P$ & $Q$.

③ Prove that $P$ & $Q$ are invertible.

<u>Comment</u>: $P$ & $Q$ are called change of basis matrices.

The set of all change of basis matrices for an $n$-dimensional vector space is called the general linear group $GL(n, \mathbb{F})$.

④ Prove that the matrix $M'$ of $L$ with respect to these new bases is given by

$$M' = PMQ$$

⑤ Suppose $L : V \xrightarrow[\text{linear}]{} V$ and $M, M'$ are the matrices of $L$ with respect to bases $(v_1, v_2, \dots, v_n)$, $(v_1', v_2', \dots, v_n')$. Find and prove a formula relating $M'$ and $M$.

(6) Now in addition assume that $(v_1, v_2 \ldots, v_n)$ and $(v_1', v_2', \ldots, v_n')$ are both orthonormal bases.

Suppose

$$(v_1' \ v_2' \ldots v_n') = (v_1 \ v_2 \ \ldots \ v_n)\, O.$$

Show

$$\begin{pmatrix} \langle v_1', v_1\rangle & \langle v_2', v_1\rangle & \cdots & \langle v_n', v_1\rangle \\ \langle v_1', v_2\rangle & \langle v_2', v_2\rangle & \cdots & \langle v_n', v_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1', v_n\rangle & \langle v_2', v_n\rangle & \cdots & \langle v_n', v_n\rangle \end{pmatrix} O = I$$

(7) For a matrix $M = (m^i_j)$ define $\tilde{m}^i_j = (m^j_i)^*$ and $M^\dagger = (\tilde{m}^i_j)$

Show $$O O^\dagger = I$$