Math 67A Homework 2 Solutions

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4.3 For each of the following sets, either show that the set is a subspace of $\mathcal{C}(\mathbb{R})$ or explain why it is not a subspace.

(a) The set $\{f \in \mathcal{C}(\mathbb{R}) | f(x) \leq 0, \forall x \in \mathbb{R}\}$

(b) The set $\{f \in \mathcal{C}(\mathbb{R}) | f(x) = 0\}$

(c) The set $\{f \in \mathcal{C}(\mathbb{R}) | f(x) = 2\}$

(d) The set of all constant functions.

(e) The set $\{\alpha + \beta \sin(x) | \alpha, \beta \in \mathbb{R}\}$.

Solution (a) This set, which I name S is not a subspace, it is not closed under scalar multiplication. Consider the constant function $f(x) = -1$ and multiply it by -1, then $-f(x) = 1$, so $f-(x) > 0$ and $-f$ is not in S.

(b) This set, which I also name S, is a subspace of $\mathcal{C}(\mathbb{R})$. I must check additive and multiplicative closure. Take $\lambda \in \mathbb{R}$ and $f \in S$ then

$$\begin{aligned} \lambda f(0) =& \lambda \cdot 0 \\ =& 0, \end{aligned}$$

so $\lambda f \in S$. Take $f, g \in S$, then

$$\begin{aligned} f(0) + g(0) =& 0 + 0 \\ =& 0, \end{aligned}$$

so $f + g \in S$. Hence, $S \subset \mathcal{C}(\mathbb{R})$ is closed under vector addition and multiplication and is a vector subspace.

(c) This set, again named S, is not a subspace. Take $f, g \in S$ then $f(1) + g(1) = 4$ so $f + g \notin S$.

(d) This set, S, is a subspace of $\mathcal{C}(\mathbb{R})$. Note that f is a constant function if and only if $f'(x) = 0$. Let $f, g \in S$, then

$$\begin{aligned} (f + g)' =& f' + g' \\ =& 0, \end{aligned}$$

so S is closed under addition. Now take $f \in S$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned} (\lambda f)' =& \lambda f' \\ =& 0, \end{aligned}$$

so S is closed under multiplication; hence, S is a subspace of $\mathcal{C}(\mathbb{R})$.

(e) This set is a vector space. Let $v = \alpha + \beta\sin(x)$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}\lambda(\alpha + \beta\sin(x)) =& \lambda\alpha + \lambda\beta\sin(x)\\ =& \alpha_1 + \beta_2\sin(x),\end{aligned}$$

so this set is closed under multiplication. If I consider the sum of two elements

$$\alpha_1 + \beta_1\sin(x) + \alpha_2 + \beta_2\sin(x) = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)\sin(x),$$

so this set is closed under vector addition; hence, this subset of $\mathcal{C}(\mathbb{R})$ is a subspace.

4.4 Give an example of a non-empty subset $U \subset \mathbb{R}^2$ such that U is closed under scalar multiplication but is not a subspace of $\mathbb{R}^2$

Solution Define U to be the union of the x and y-axes; i.e., all points with x or y component zero. This set is closed under scalar multiplication because the product of any number and zero is zero; thus if the x component is zero this property is maintained by scalar multiplication, and likewise if the y component is zero. This set is not closed under addition as $(0,1) + (1,0) = (1,1) \notin U$.

4.5 Let $\mathbb{F}[z]$ denote the vector space of all polynomials having coefficients over $\mathbb{F}$, and define U to be the subspace of $\mathbb{F}[z]$ given by

$$U = \{az^2 + bz^5 | a, b \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}$ such that $\mathbb{F}[z] = U \oplus W$

Solution Let W be the set of all polynomials in z over $\mathbb{F}$ with no degree two or degree five term. To show that $\mathbb{F}[z]$ is a direct sum of U, W I must show that any element of $\mathbb{F}[z]$ can be written as the sum of an element in U, W. Let p be a polynomial, then p had finite degree n and may be written as

$$\sum_{i=0}^{n} a_i z^i + \sum_{i=n+1}^{\infty} 0z^i = a_2z^2 + a_5z^5 + \underbrace{\sum_{i=0}^{1} a_i z^i + \sum_{i=3}^{4} a_i z^i + \sum_{i=6}^{n} a_i z^i}_{q},$$

where $q \in W$. The second requirement is that this representation must be unique. It is sufficient to check that the representation of the zero vector is unique. The zero polynomial has zero as its coefficient for all terms, which is uniquely written as the sum of the zero of U and the zero of W.

PWE 4.2 Let V be a vector space over $\mathbb{F}$. The, given $a \in \mathbb{F}$ and $v \in V$ such that $av = 0$, prove that either $a = 0$ or $v = 0$.

Solution Note that if $a = 0$ or $v = 0$ then $av = 0$. What remains to be proven is that if $av = 0$ then one of a, v must be zero. Suppose, towards a contradiction, that $av = 0$ and neither a nor v were zero. Since $a \neq 0$ there exists a^{-1}, then

$$\begin{aligned}a^{-1}(av) =& v\\ \neq& 0;\end{aligned}$$

however,

$$\begin{aligned}a^{-1}(av) =& a^{-1}0\\ =& 0,\end{aligned}$$

which is a contradiction. Thus, it must be that one of a, v was zero. This completes the proof.

PWE 4.3 Prove or give a counterexample to the following claim:

Claim. Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2, and W_3 are subspace of V such that $W_1 + W_3 = W_2 + W_3$, then $W_1 = W_2$.

Solution The claim is false. Let $V = \mathbb{F}^2$, and $W_3 = V$ while $W_1 = \text{span}(0,1)$ and $W_2 = \text{span}(1,0)$. Then $W_1 + W_3 = V = W_2 + W_3$ but $W_1 \neq W_2$.

PWE 4.4 Prove or give a counterexample to the following claim:

Claim. Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2, and W_3 are subspace of V such that $W_1 \oplus W_3 = W_2 \oplus W_3$, then $W_1 = W_2$.

Solution This is also false. Again take $V = \mathbb{F}^2$, but this time take $W_1 = \text{span}(1,0)$,$W_2 = \text{span}(0,1)$, and $W_3 = \text{span}(1,1)$.

5.1 Show that the vectors $v_1 = (1,1,1)$, $v_2 = (1,2,3)$, and $v_3 = (2,-1,1)$ are linearly independent in $\mathbb{R}^3$. Write $(1,-2,5)$ as a linear combination of v_1, v_2 and v_3.

Solution The simplest way to see if three vectors are linearly dependent or independent and to decompose a vector into a sum of those vectors is through Gauss-Jordan elimination on an augmented matrix

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

so v_1, v_2 and v_3 are linearly independent and $-6v_1 + 3v_2 + 2v_3 = (1,-2,5)$

5.2 Consider the complex vector space $V = \mathbb{C}^3$ and the list (v_1, v_2, v_3) of vectors in V, where

$$v_1 = (i,0,0), \qquad v_2 = (i,1,0), \qquad v_3 = (i,i,-1).$$

(a) Prove that $\text{span}(v_1, v_2, v_3) = V$.

(b) Prove or disprove: (v_1, v_2, v_3) is a basis for V.

Solution (a) Let $(a_1, a_2, a_3) \in \mathbb{C}^3$. Gauss-Jordan elimination is reliable method of finding the necessary coefficients to represent this vector in terms of v_1, v_2, v_3

$$\begin{bmatrix} i & i & i & a_1 \\ 0 & 1 & i & a_2 \\ 0 & 0 & -1 & a_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & -ia_1 \\ 0 & 1 & i & a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -ia_1 + a_3 \\ 0 & 1 & 0 & a_2 + ia_3 \\ 0 & 0 & 1 & -a_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -ia_1 - a_2 + (1-i)a_3 \\ 0 & 1 & 0 & a_2 + ia_3 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}.$$

This computation shows that $(a_1, a_2, a_3) = (-ia_1 - a_2 + (1-i)a_3)v_1 + (a_2 + ia_3)v_2 - a_3v_3$; thus any element of $\mathbb{C}^3$ can be written as a linear combination of v_1, v_2, v_3 so $\text{span}(v_1, v_2, v_3) = V$.

(b) The set $\{v_1, v_2, v_3\}$ is a basis for V. A basis is a linearly independent spanning set. In part (a) I showed that this set spans V. Since the row reduced matrix had all ones along the main diagonal we also know that v_1, v_2, v_3 are linearly independent, thus they are a basis of V.

PWE 5.1 Let V be a vector space over $\mathbb{F}$, and suppose that the list $(v_1, v_2, ..., v_n)$ of vectors spans V, where each $v_i \in V$. Prove that the list

$$(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$$

also spans V.

Solution Choose $v \in V$. Since $(v_1, v_2, ..., v_n)$ spans V we have that for some $a_i \in \mathbb{F}$

$$w = a_1v_1 + ... + a_nv_n.$$

I seek a way to write this in terms of the second set of vectors $(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$ and see that $-v_k = (v_1 - v_2) + (v_2 - v_3) + ... + (v_{k-1} - v_k)$; thus,

$$w = a_1(v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + ... + (\sum_{i=1}^{n} a_i)v_n.$$

The vector w was arbitrary, so any element of V can be written as a linear combination of $(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$; thus $(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$ spans V.