Name: _____

QUIZ 4

Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across the ellipse

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (4\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

We have

$$\frac{d\mathbf{r}}{dt} = (-3\sin t)\,\mathbf{i} + (4\cos t)\,\mathbf{j},$$

$$\mathbf{F}_1 = (3\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j}, \quad \text{and}$$

$$\mathbf{F}_2 = (-4\sin t)\,\mathbf{i} + (3\cos t)\,\mathbf{j}.$$

So

$$\mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = -9\cos t\sin t + 16\cos t\sin t = 7\cos t\sin t \quad \text{and}$$

$$\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 12\sin^2 t + 12\cos^2 t = 12.$$

Then

$$\text{Circ}_1 = \int_0^{2\pi} \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} 7\cos t\sin t dt = \frac{7}{2}\left[\sin^2 t\right]_0^{2\pi} = 0,$$

$$\text{Circ}_2 = \int_0^{2\pi} \mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} 12dt = 24\pi,$$

$$\text{Flux}_1 = \int_0^{2\pi} (M_1 dy - N_1 dx) = \int_0^{2\pi} ((3\cos t)(4\cos t)\,dt - (4\sin t)(-3\sin t)\,dt)$$

$$= \int_0^{2\pi} 12dt = 24\pi, \quad \text{and}$$

$$\text{Flux}_2 = \int_0^{2\pi} (M_2 dy - N_2 dx) = \int_0^{2\pi} ((-4\sin t)(4\cos t)\,dt - (3\cos t)(-3\sin t)\,dt)$$

$$= -\int_0^{2\pi} 7\cos t\sin t dt = 0.$$

There is another way to calculate the fluxes, but in this case it is more complicated. Since $|d\mathbf{r}/dt| = 5$,

$$\mathbf{T} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|} = \left(-\frac{3}{5}\sin t\right)\mathbf{i} + \left(\frac{4}{5}\cos t\right)\mathbf{j}, \quad \text{and}$$

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{5}\sin t & \frac{4}{5}\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(\frac{4}{5}\cos t\right)\mathbf{i} + \left(\frac{3}{5}\sin t\right)\mathbf{j}.$$

So

$$\mathbf{F}_1 \cdot \mathbf{n} = \frac{12}{5}\cos^2 t + \frac{12}{5}\sin^2 t = \frac{12}{5} \quad \text{and}$$

$$\mathbf{F}_2 \cdot \mathbf{n} = \frac{-16}{5}\cos t \sin t + \frac{9}{5}\cos t \sin t = \frac{-7}{5}\cos t \sin t.$$

Just like before,

$$\text{Flux}_1 = \int_0^{2\pi} \mathbf{F}_1 \cdot \mathbf{n} \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_0^{2\pi} 12 dt = 24 \quad \text{and}$$

$$\text{Flux}_2 = \int_0^{2\pi} \mathbf{F}_2 \cdot \mathbf{n} \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_0^{2\pi} -7\cos t \sin t dt = 0.$$