Name: ____________________

QUIZ 5

Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (away from the z axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$. Use any method you wish, but show your work.

Method from 16.5: We will project onto the x-y plane, so $\mathbf{p} = \mathbf{k}$. Since the surface is cut below $z = 1$, we have $z = x^2 + y^2 \leq 1$ as our region of integration. The surface is defined by $f(x, y, z) = x^2 + y^2 - z = 0$. This gives $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ and $|\nabla f \cdot \mathbf{p}| = |-1| = 1$. Since ∇f has positive $\mathbf{i}$- and $\mathbf{j}$-components in the first octant, it points in the direction of the required normal vector $\mathbf{n}$ (away from the z-axis). So now, with $\mathbf{F} \cdot \nabla f = 8x^2 + 8y^2 - 2$, we can use formula (10) from page 1172 of the text:

$$\text{Flux} = \int\int_R \mathbf{F} \cdot \frac{\nabla f}{|\nabla f \cdot \mathbf{p}|} dA = \int\int_{x^2 + y^2 \leq 1} \left(8x^2 + 8y^2 - 2\right) dA$$

$$= \int_0^{2\pi} \int_0^1 \left(8r^2 - 2\right) r dr d\theta = \int_0^{2\pi} \left[2r^4 - r^2\right]_0^1 d\theta = 2\pi$$

Method from 16.6: We can use polar coordinates as our parameterization. Then $x = u \cos v$, $y = u \sin v$, and $z = x^2 + y^2 = u^2$, with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$, so the surface is $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k}$, with $\mathbf{r}_u = \cos v\mathbf{i} + \sin v\mathbf{j} + 2u\mathbf{k}$ and $\mathbf{r}_v = -u \sin v\mathbf{i} + u \cos v\mathbf{j}$. Now $\mathbf{F} = 4u \cos v\mathbf{i} + 4u \sin v\mathbf{i} + 2\mathbf{k}$ and $\mathbf{r}_u \times \mathbf{r}_v = -2u^2 \cos v\mathbf{i} - 2u^2 \sin v\mathbf{j} + u\mathbf{k}$, so $\mathbf{F} \cdot (-\mathbf{r}_u \times \mathbf{r}_v) = 8u^3 - 2u$. (We have to use the opposite of $\mathbf{r}_u \times \mathbf{r}_v$ so that it points in the correct direction for the normal $\mathbf{n}$.) With this,

$$\text{Flux} = \int\int_R \mathbf{F} \cdot (-\mathbf{r}_u \times \mathbf{r}_v)\, dA = \int_0^{2\pi} \int_0^1 \left(8u^3 - 2u\right) du dv,$$

which is the same integral as before.