16A Section 1 Exam 2
May 6, 2009

Your Name: Key

Your Student ID:

1.) It is a violation of the university honor code to, in any way, assist another person in the completion of this exam, copy answers from another student's exam, have another student take your exam for you. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
2.) No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.
3.) Using only a calculator to determine the value of limits will receive little credit.
4.) You will be graded on proper use of the derivative and limit notation.
5.) Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.
6.) The following trigonometry identities are at your disposal:
(a) $\sin 2\theta = 2\sin\theta\cos\theta$
(b) $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

Problem	1	2	3	4	5	Bonus	Total
Points	18	20	20	22	20	10	100 + 10
Score							

Problem 1 [18 points]

(a) [6 points] Use the power rule to differentiate the function $f(x) = \frac{1}{x^2}$.
Ans: Rewriting $f(x) = x^{-2}$, we have $f'(x) = (-2)x^{-2-1} = -2x^{-3}$, which could also be written as $f'(x) = \frac{-2}{x^3}$.

(b) [6 points] Use the power rule to differentiate the function $f(x) = \sqrt{x}$
Ans:
rewrite: $f(x) = x^{1/2}$
so

$$f'(x) = 1/2 \cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$$

(c) [6 points] Differentiate $y = f(x) = \sqrt{2x+3}$ (Hint: power rule and chain rule)
Ans:
Let $u = 2x + 3$, then $y = u^{1/2}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

$$= \frac{1}{2}u^{-1/2} \cdot 2 = \frac{1}{2} \cdot (2x+3)^{-1/2} \cdot 2 \tag{2}$$

$$= \frac{2}{2\sqrt{2x+3}} \tag{3}$$

$$= \frac{1}{\sqrt{2x+3}} \tag{4}$$

Problem 2 [20 points]

(a) [6 points] A right triangle has one angle $\theta = 30^o$, and its adjacent side has length 100, its opposite side has length y.

Ans: $y = \tan(30^o) * 100 = 100 \times \frac{\sqrt{3}}{3} = \frac{100\sqrt{3}}{3}$

(b) [6 points] Find the period of the function

$$y = 5 \tan\left(\frac{2\pi x}{3}\right)$$

Ans:

period is

$$\frac{\pi}{\frac{2\pi}{3}} = \frac{3}{2}$$

(c) [8 points] Solve the following trigonometry equation for $0 \leq \theta \leq 2\pi$:

$$4\,(\sin\theta)^2 = 3$$

Ans:

Using the trig identity (b) on page 1, we have

$$1 - \cos 2\theta = 2 * \sin^2\theta = 2 * (\sin\theta)^2$$

or

$$\frac{1 - \cos 2\theta}{2} = (\sin\theta)^2\,,$$

which allows us to rewrite the equation given in the problem as

$$4 * \frac{1 - \cos 2\theta}{2} = 3$$

or

$$\frac{1 - \cos 2\theta}{2} = 3/4$$

or

$$1 - \cos 2\theta = 3/2$$

or

$$\cos 2\theta = -1/2$$

which gives $2\theta = \frac{2\pi}{3}$ or $2\theta = \frac{4\pi}{3}$ or $2\theta = \frac{8\pi}{3}$ or $2\theta = \frac{10\pi}{3}$. Therefore, $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

Problem 3 [20 points]

Differentiate each of the following functions. Do Not Simplify Answers.

(a) [5 points] $y = 3x^2 + x^{-7} - \frac{15}{23}$

Ans:

$$\begin{aligned} y' &= 6x + (-7) \cdot x^{-8} && (5) \\ &= 6x - \frac{7}{x^8} && (6) \end{aligned}$$

(b) [5 points] $f(x) = x^{100} \cdot (2x^2 + 5)$

Ans:

$$\begin{aligned} f'(x) &= (x^{100})' \cdot (2x^2 + 5) + x^{100} \cdot (2x^2 + 5)' && (7) \\ &= 100 \cdot x^{99} \cdot (2x^2 + 5) + x^{100} \cdot 4x && (8) \end{aligned}$$

(c) [5 points] $f(x) = \frac{x-2}{3-x^5}$

Ans:

$$\begin{aligned} f'(x) &= \frac{(x-2)' \cdot (3-x^5) - (3-x^5)' \cdot (x-2)}{(3-x^5)^2} && (9) \\ &= \frac{1 \cdot (3-x^5) - (-5)x^4 \cdot (x-2)}{(3-x^5)^2} && (10) \\ &= \frac{(3-x^5) + 5x^4 \cdot (x-2)}{(3-x^5)^2} && (11) \end{aligned}$$

(d) [5 points] $f(x) = (\tan x)^3$

Ans:

$$\begin{aligned} f'(x) &= 3(\tan x)^2 \cdot \frac{d}{dx}[\tan x] && (12) \\ &= 3(\tan x)^2 \cdot (\sec x)^2 && (13) \\ & && (14) \end{aligned}$$

Problem 4 [22 points]

(a) [8 points] Find the period and amplitude of the function $y = 2\sin(\pi x)$.
Ans:
Period: $T = \frac{2\pi}{\pi} = 2$
Amplitude: 2
(b) [6 points] Find the limit of the following trigonometric function.

$$\lim_{x \to 0} \frac{\sin(2x)}{2x}$$

(Hint: use either graphic calculator or the replacement theorem)
Ans:
Let $u = 2x$, we can rewrite the given limit as

$$\lim_{u \to 0} \frac{\sin(u)}{u},$$

which has the limit 1 as u approaches 0. So

$$\lim_{u \to 0} \frac{\sin(u)}{u} = 1.$$

(c) [8 points] Find the first and second derivatives of the function $y = \frac{2}{x}$.
Ans:

$$f'(x) = \frac{d}{dx}\left[2x^{-1}\right] = 2 \cdot (-1) \cdot x^{-2} = -2 \cdot x^{-2}$$
$$f''(x) = \frac{d}{dx}\left[-2 \cdot x^{-2}\right] = (-2) \cdot (-2) \cdot x^{-3} = 4 \cdot x^{-3}$$

Problem 5 [20 points]

Find the derivative of each of the following functions.

(a) [5 points]

$$f(x) = 2\cos\theta \cdot \sin\theta$$

Ans:

$$f(x) = \sin 2\theta \quad (15)$$

so

$$f'(x) = 2\cos 2\theta \quad (16)$$

(b) [5 points]

$$f(x) = \tan(x^2)$$

Ans:

$$\begin{aligned} f'(x) &= (\sec(x^2))^2 \cdot 2x && (17) \\ &= 2x(\sec(x^2))^2 && (18) \end{aligned}$$

(c) [5 points]

$$f(x) = \frac{1}{2}x^2 \cdot \sin x$$

Ans:

$$\begin{aligned} f'(x) &= (\frac{1}{2}x^2)' \cdot \sin x + \frac{1}{2}x^2 \cdot (\sin x)' && (19) \\ &= x \cdot \sin x + \frac{1}{2}x^2 \cdot \cos x && (20) \end{aligned}$$

(d) [5 points]

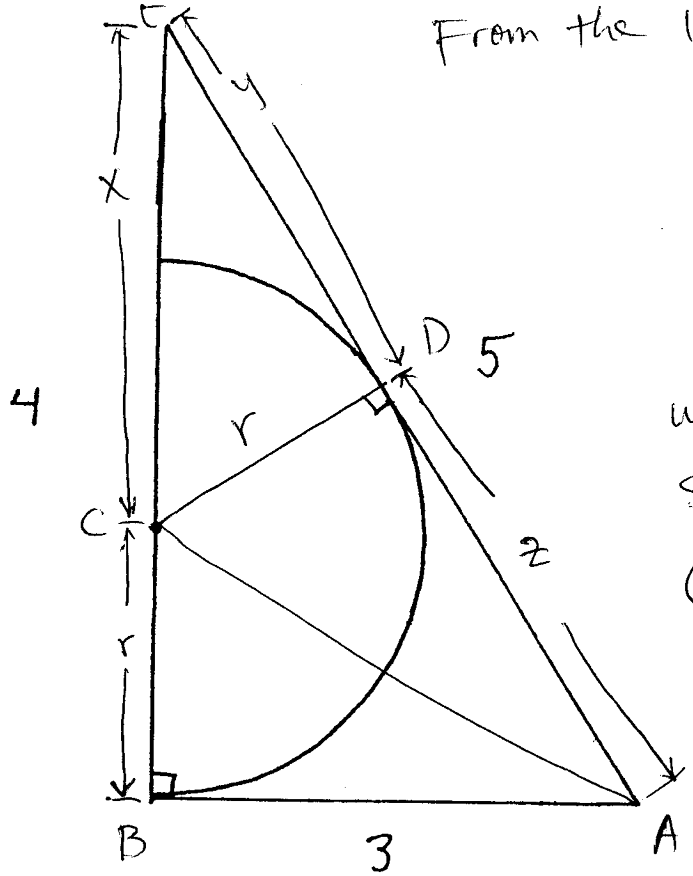
$$f(x) = \sqrt{\sin x}$$

Ans:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[(\sin x)^{-\frac{1}{2}}\right] && (21) \\ &= -\frac{1}{2} \cdot (\sin x)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}[\sin x] && (22) \\ &= -\frac{1}{2} \cdot (\sin x)^{-\frac{3}{2}} \cdot \cos x && (23) \end{aligned}$$

EXTRA CREDIT PROBLEM– The following problem is worth 10 points. This problem is OPTIONAL.

1.) Determine the radius r of the semi-circle which is inscribed in the given right triangle.



From the left, we know

$$y^2 + r^2 = x^2$$

$$y + z = 5$$

$$x + r = 4$$

we also know that $z = 3$ since the two triangles (right angled) ABC and ADC are the same (i.e they both share a common edge $\overline{AC}$, and have the shorter edge of length r ($\overline{BC}$ and $\overline{DC}$))

so $y = 5 - 3 = 2$,

Using the triangle CDE, we have

$$x^2 = r^2 + y^2, \quad \text{or} \quad (4 - r)^2 = r^2 + 2^2$$

$$\Rightarrow 16 - 8r + r^2 = r^2 + 4 \Rightarrow 8r = 12$$

$$\Rightarrow r = \frac{3}{2}$$