16A Xiao Exam 3
May 22, 2009

Your Name:

Your Student ID:

Key

1.) It is a violation of the university honor code to, in any way, assist another person in the completion of this exam, copy answers from another student's exam, have another student take your exam for you. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
2.) No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.
3.) Using only a calculator to determine the value of limits will receive little credit.
4.) You will be graded on proper use of the derivative notation.
4.) You will be graded on the neatness when you do the detailed graphing of functions.
5.) Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam.
6.) Include units on answers where units are appropriate.
7.) Make sure you have 6 pages including the cover page.

Problem	1	2	3	4	5	Bonus	Total
Points	20	20	20	20	20	20	100 + 10
Score							

Problem 1 [20 points]

Find $\frac{dy}{dx}$ assuming x and y is related by each equation below. Do NOT Simplify Answers.

(a) [10 points] $x^2 + y^2 = 4$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d}{dx}\left[(xy)^{\frac{1}{2}} \right] = \frac{d}{dx}[4]$$

$$\Rightarrow \quad \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot \left(y + x\frac{dy}{dx} \right) = 0$$

$$\Rightarrow \quad y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

Problem 2 [20 points]

(a) [10 points] Analytically find the intervals on which the graph of the function below is concave upward and those on which it is concave downward. DO NOT GRAPH THE FUNCTION.

$$y = (x-1)^3(x-5)$$

$y' = 3(x-1)^2(x-5) + (x-1)^3 \cdot 1$

$= (x-1)^2(3x-15+x-1) = 4(x-1)^2(x-4)$

$y'' = 8(x-1)\cdot(x-4) + 4(x-1)^2$

$= 4(x-1)(2x-8+x-1) = 4(x-1)(3x-9)$

$= 12(x-1)(x-3)$

$y'' = 0 \Rightarrow x = 1$ or $x = 3$

$+$ | 1 | $-$ | 3 | $+$ $\quad y'$

y is $\cup$ for $x > 3$ or $x < 1$; y is $\cap$ for $1 < x < 3$

(b) [10 points] Determine all inflection points (x, y) for the graph of $f(x) = \sin x - \cos x$ on the interval $[0, 2\pi]$. DO NOT GRAPH THE FUNCTION.

$f'(x) = \cos x + \sin x$

$f''(x) = -\sin x + \cos x$

$f''(x) = 0 \Rightarrow -\sin x + \cos x = 0$

$\Rightarrow \sin x = \cos x \Rightarrow \dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\cos x}$

$\Rightarrow \tan x = 1$

$\Rightarrow x = \dfrac{\pi}{4}$ or $x = \dfrac{5\pi}{4}$

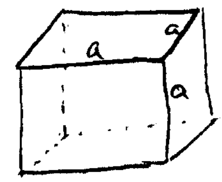
$+$ | $\frac{\pi}{4}$ | $-$ | $\frac{5\pi}{4}$ | $+$ $\quad f''$

inflection points: $x = \frac{1}{4}\pi$, $x = \dfrac{5\pi}{4}$

Problem 3 [20 points]

(a) [10 points] The volume of a cube is increasing at a rate of 3 in^3/sec. How fast is the surface area changing when each edge is 2 inches long?

$V = a^3$



$S = 6a^2$

$\frac{dV}{dt} = 3 \text{ in}^3/\text{sec}$

$\Rightarrow 3 = 3a^2 \frac{da}{dt} \Rightarrow \frac{da}{dt} = \frac{1}{a^2}$

$\frac{dS}{dt} = 12a \cdot \frac{da}{dt}$

So at $a = 2$ inch, $\frac{dS}{dt} = 12 \cdot 2 \cdot \frac{1}{2^2} = 6 \text{ in}^2/\text{sec}$

(b) [10 points] Assuming that y is a function of x and that $y^3 - xy = 4$. Find an equation of the line perpendicular to the graph of this equation at $x = 0$.

Taking implicit derivative:

$\frac{d}{dx}[y^3 - xy] = \frac{d}{dx}[4] \Rightarrow 3y^2 \frac{dy}{dx} - y - x\frac{dy}{dx} = 0$

$\Rightarrow (3y^2 - x)\frac{dy}{dx} = y \Rightarrow \frac{dy}{dx} = \frac{y}{3y^2 - x}$

At $x = 0$, y satisfies: $y^3 - 0 \cdot y = 4 \Rightarrow y = \sqrt[3]{4}$

So the slope of the tangent line of the function is: $m = \frac{\sqrt[3]{4}}{3(\sqrt[3]{4})^2 - 0} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{4}}$

slope of the perpendicular line is:

$m_1 = -\frac{1}{m} = -3 \cdot \sqrt[3]{4}$

equation of the line is

$y - \sqrt[3]{4} = -3 \cdot \sqrt[3]{4}\,(x - 0) \Rightarrow y = \sqrt[3]{4} + 3 \cdot \sqrt[3]{4}\, x$

Problem 4 [20 points]

(a) [10 points] For the following function f, state the domain and determine all absolute and relative minimum and maximum values, inflection points, and $x-$ and $y-$intercepts. State clearly the x-values for which f is increasing ($\uparrow$), decreasing ($\downarrow$), concave up ($\bigcup$), concave down ($\bigcap$).

$$f(x) = \frac{x^2+1}{x^2-2}$$

The domain is: $X \neq \sqrt{2}$, $X \neq -\sqrt{2}$

$$f'(x) = \frac{2X(X^2-2) - (X^2+1)(2X)}{(X^2-2)^2} = \frac{-6X}{(X^2-2)^2}$$

$f'(x) = 0 \Rightarrow X = 0$

f': $+$ | $+$ | $-$ | $-$ at $X=-\sqrt{2}$, $X=0$, $X=\sqrt{2}$

$X=0$, $Y=-\frac{1}{2}$ rel. maximum

$f(x)$ is $\uparrow$ when $X<0$ ($X \neq -\sqrt{2}$)

$f(x)$ is $\downarrow$ when $X>0$ ($X \neq \sqrt{2}$)

$$f''(x) = \frac{-6\cdot(X^2-2)^2 - (-6X)\cdot 2(X^2-2)\cdot 2X}{(X^2-2)^4}$$

$$= \frac{(X^2-2)(-6X^2+12+24X^2)}{(X^2-2)^4} = \frac{18X^2+12}{(X^2-2)^3} = \frac{6(3X^2+2)}{(X^2-2)^3}$$

$f''(X) = 0 \Rightarrow 6\cdot(3X^2+2) = 0 \Rightarrow$ no solution.

f'': $+$ | $-$ | $+$ at $-\sqrt{2}$, $\sqrt{2}$

no inflection point

$f(x)$ is $\bigcup$ for $X>\sqrt{2}$ or $X<-\sqrt{2}$

$f(x)$ is $\bigcap$ for $-\sqrt{2}<X<\sqrt{2}$

Y-intercepts: $X=0$, $Y=-\frac{1}{2}$

X-intercepts: none

Problem 4 [20 points]

(a) [10 points] For the following function f, state the domain and determine all absolute and relative minimum and maximum values, inflection points, and $x-$ and $y-$intercepts. State clearly the x-values for which f is increasing ($\uparrow$), decreasing ($\downarrow$), concave up ($\bigcup$), concave down ($\bigcap$).

$$f(x) = \frac{x^2+1}{x^2-2}$$

see previous page

(b) [10 points] Find two positive numbers whose sum is 10 and whose product is a maximum.

$x+y=10$ $\quad$ $x>0$

$P = xy$ $\quad$ $y>0$

$y = 10-x$

so $P = x(10-x)$

$P' = 1\cdot(10-x) + x(-1)$

$= 10 - x - x = 10-2x$

$P' = 0 \Rightarrow 10-2x = 0 \Rightarrow x = 5 > 0$

$P'' = -2$, $P''(5) = -2 < 0$

so $x=5$ is relative max and absolute max

Problem 5 (Multiple-Choice and True-and-False Questions) [20 points]

(1) The second derivative of $f(x) = x + 100$ is:
(a) $f''(x) = 0$ (b) $f''(x) = 1$ (c) $f''(x) = 100$ (d) $f''(x)$ is undefined

(2) The domain of $f(x) = 1/(x^2 - 1)$ is
(a) all real numbers (b) all real numbers except for $x = 1$ or $x = -1$
(c) $-1 \leq x \leq 1$ (d) $-1 < x < 1$

(3) If

$$f'(x) = 2 - \frac{2}{x^{1/3}} = \frac{2(x^{1/3} - 1)}{x^{1/3}} = 0,$$

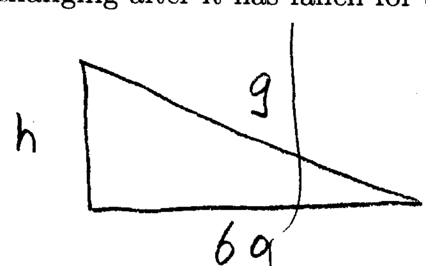
we must have
(a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) no solution

(4) The second derivative represents the rate of change of the first derivative.
■ True ☐ False

(5) If f has a relative minima when $x = c$, then it must be that either $f'(c) = 0$ or $f'(c)$ is undefined.
■ True ☐ False

Bonus Problem [10 points]
The problem below is optional, and worths 10 points.

A watermelon is dropped from a height of 224 feet. It will strike the ground 60 feet from where you are standing. How fast is the distance between you and the watermelon changing after it has fallen for 3 seconds?



$$h = -16t^2 + 224$$

$$g^2 = h^2 + 60^2$$

$$\frac{dh}{dt} = -32t$$

$$2g\frac{dg}{dt} = 2h \cdot \frac{dh}{dt} + 0$$

$$\Rightarrow \frac{dg}{dt} = \frac{h \cdot dh/dt}{g}$$

At $t = 3$ sec, $h = -16 \cdot 9 + 224 = 80$ feet

$g = \sqrt{60^2 + 80^2} = 100$ feet

$dg/dt = \frac{80 \cdot (-32 \times 3)}{\ldots} = -76.8$ feet/se