Problem 1 [18 points]

(a) [6 points] Use the power rule to differentiate the function $f(x) = \frac{1}{x^2}$.
Ans: Rewriting $f(x) = x^{-2}$, we have $f'(x) = (-2)x^{-2-1} = -2x^{-3}$, which could also be written as $f'(x) = \frac{-2}{x^3}$.

(b) [6 points] Use the power rule to differentiate the function $f(x) = \sqrt{x}$
Ans:
rewrite: $f(x) = x^{1/2}$
so

$$f'(x) = 1/2 \cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$$

(c) [6 points] Differentiate $y = f(x) = \sqrt{2x+3}$ (Hint: power rule and chain rule)
Ans:
Let $u = 2x + 3$, then $y = u^{1/2}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (1) \\
&= \frac{1}{2}u^{-1/2} \cdot 2 = \frac{1}{2} \cdot (2x+3)^{-1/2} \cdot 2 && (2) \\
&= \frac{2}{2\sqrt{2x+3}} && (3) \\
&= \frac{1}{\sqrt{2x+3}} && (4)
\end{aligned}$$

Problem 2 [20 points]

(a) [6 points] A right triangle has one angle $\theta = 30^o$, and its adjacent side has length 100, its opposite side has length y.

Ans: $y = \tan(30^o) * 100 = 100 \times \frac{\sqrt{3}}{3} = \frac{100\sqrt{3}}{3}$

(b) [6 points] Find the period of the function

$$y = 5\ \tan\left(\frac{2\pi x}{3}\right)$$

Ans:

period is

$$\frac{\pi}{\frac{2\pi}{3}} = \frac{3}{2}$$

(c) [8 points] Solve the following trigonometry equation for $0 \leq \theta \leq 2\pi$:

$$4\,(\sin\theta)^2 = 3$$

Ans:

Using the trig identity (b) on page 1, we have

$$1 - \cos 2\theta = 2 * \sin^2\theta = 2 * (\sin\theta)^2$$

or

$$\frac{1-\cos 2\theta}{2} = (\sin\theta)^2\,,$$

which allows us to rewrite the equation given in the problem as

$$4 * \frac{1-\cos 2\theta}{2} = 3$$

or

$$\frac{1-\cos 2\theta}{2} = 3/4$$

or

$$1 - \cos 2\theta = 3/2$$

or

$$\cos 2\theta = -1/2$$

which gives $2\theta = \frac{2\pi}{3}$ or $2\theta = \frac{4\pi}{3}$ or $2\theta = \frac{8\pi}{3}$ or $2\theta = \frac{10\pi}{3}$. Therefore, $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

Problem 3 [20 points]

Differentiate each of the following functions. Do Not Simplify Answers.

(a) [5 points] $y = 3x^2 + x^{-7} - \frac{15}{23}$

Ans:

$$\begin{aligned} y' &= 6x + (-7) \cdot x^{-8} && (5) \\ &= 6x - \frac{7}{x^8} && (6) \end{aligned}$$

(b) [5 points] $f(x) = x^{100} \cdot (2x^2 + 5)$

Ans:

$$\begin{aligned} f'(x) &= (x^{100})' \cdot (2x^2 + 5) + x^{100} \cdot (2x^2 + 5)' && (7) \\ &= 100 \cdot x^{99} \cdot (2x^2 + 5) + x^{100} \cdot 4x && (8) \end{aligned}$$

(c) [5 points] $f(x) = \frac{x-2}{3-x^5}$

Ans:

$$\begin{aligned} f'(x) &= \frac{(x-2)' \cdot (3 - x^5) - (3 - x^5)' \cdot (x-2)}{(3-x^5)^2} && (9) \\ &= \frac{1 \cdot (3 - x^5) - (-5)x^4 \cdot (x-2)}{(3-x^5)^2} && (10) \\ &= \frac{(3 - x^5) + 5x^4 \cdot (x-2)}{(3-x^5)^2} && (11) \end{aligned}$$

(d) [5 points] $f(x) = (\tan x)^3$

Ans:

$$\begin{aligned} f'(x) &= 3(\tan x)^2 \cdot \frac{d}{dx}[\tan x] && (12) \\ &= 3(\tan x)^2 \cdot (\sec x)^2 && (13) \\ & && (14) \end{aligned}$$

Problem 4 [22 points]

(a) [8 points] Find the period and amplitude of the function $y = 2\sin(\pi x)$.
Ans:
Period: $T = \frac{2\pi}{\pi} = 2$
Amplitude: 2
(b) [6 points] Find the limit of the following trigonometric function.

$$\lim_{x \to 0} \frac{\sin(2x)}{2x}$$

(Hint: use either graphic calculator or the replacement theorem)
Ans:
Let $u = 2x$, we can rewrite the given limit as

$$\lim_{u \to 0} \frac{\sin(u)}{u},$$

which has the limit 1 as u approaches 0. So

$$\lim_{u \to 0} \frac{\sin(u)}{u} = 1.$$

(c) [8 points] Find the first and second derivatives of the function $y = \frac{2}{x}$.
Ans:

$$f'(x) = \frac{d}{dx}\left[2x^{-1}\right] = 2 \cdot (-1) \cdot x^{-2} = -2 \cdot x^{-2}$$
$$f''(x) = \frac{d}{dx}\left[-2 \cdot x^{-2}\right] = (-2) \cdot (-2) \cdot x^{-3} = 4 \cdot x^{-3}$$

Problem 5 [20 points]

Find the derivative of each of the following functions.

(a) [5 points]

$$f(x) = 2\cos\theta \cdot \sin\theta$$

Ans:

$$f(x) = \sin 2\theta \tag{15}$$

so

$$f'(x) = 2\cos 2\theta \tag{16}$$

(b) [5 points]

$$f(x) = \tan(x^2)$$

Ans:

$$\begin{aligned} f'(x) &= (\sec(x^2))^2 \cdot 2x && (17)\\ &= 2x(\sec(x^2))^2 && (18) \end{aligned}$$

(c) [5 points]

$$f(x) = \frac{1}{2}x^2 \cdot \sin x$$

Ans:

$$\begin{aligned} f'(x) &= (\frac{1}{2}x^2)' \cdot \sin x + \frac{1}{2}x^2 \cdot (\sin x)' && (19)\\ &= x \cdot \sin x + \frac{1}{2}x^2 \cdot \cos x && (20) \end{aligned}$$

(d) [5 points]

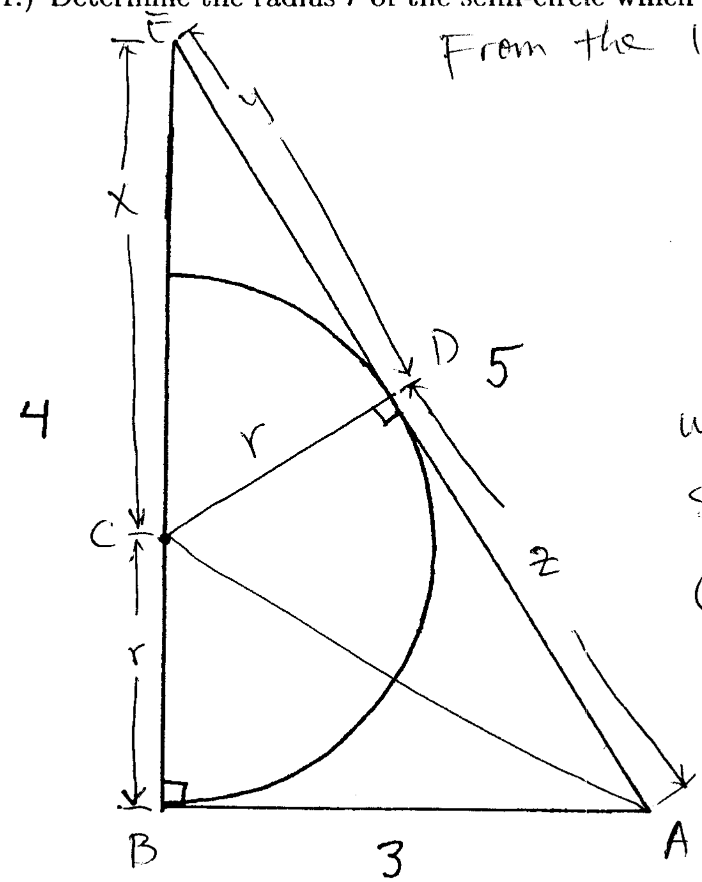
$$f(x) = \sqrt{\sin x}$$

Ans:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[(\sin x)^{+\frac{1}{2}}\right] && (21)\\ &= -\frac{1}{2} \cdot (\sin x)^{+\frac{1}{2}-1} \cdot \frac{d}{dx}[\sin x] && (22)\\ &= -\frac{1}{2} \cdot (\sin x)^{-\frac{1}{2}} \cdot \cos x && (23) \end{aligned}$$

EXTRA CREDIT PROBLEM– The following problem is worth 10 points. This problem is OPTIONAL.

1.) Determine the radius r of the semi-circle which is inscribed in the given right triangle.



From the left, we know

$$y^2 + r^2 = x^2$$

$$y + z = 5$$

$$x + r = 4$$

we also know that $z = 3$ since the two triangles (right angled) ABC and ADC are the same (i.e they share a common edge $\overline{AC}$, and both have the shorter edges of length r) ($\overline{BC}$ and $\overline{DC}$)

so $y = 5 - 3 = 2$,

Using the triangle CDE, we have

$x^2 = r^2 + y^2$, or $(4-r)^2 = r^2 + 2^2$

$\Rightarrow 16 - 8r + r^2 = r^2 + 4 \Rightarrow 8r = 12$

$\Rightarrow r = \frac{3}{2}$