

Problem 1

[10 points]

Find $\frac{dy}{dx}$ assuming x and y is related by each equation below. Do NOT Simplify Answers.

(a) [5 points] $x^2 + y^2 = 4$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

(b) [5 points] $\sqrt{xy} = 4$

$$\frac{d}{dx} \left[(xy)^{\frac{1}{2}} \right] = \frac{d}{dx} [4]$$

$$\Rightarrow \frac{1}{2} (xy)^{-\frac{1}{2}} \cdot (y + x \frac{dy}{dx}) = 0$$

$$\Rightarrow y + x \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

Problem 2

[10 points]

(a) [5 points] Analytically find the intervals on which the graph of the function below is concave upward and those on which it is concave downward. DO NOT GRAPH THE FUNCTION.

$$y = (x-1)^3(x-5)$$

$$y' = 3(x-1)^2(x-5) + (x-1)^3 \cdot 1$$

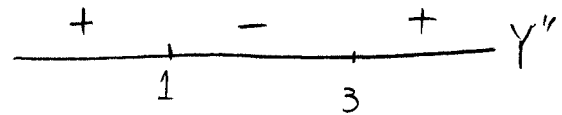
$$= (x-1)^2(3x-15+x-1) = 4(x-1)^2(x-4)$$

$$y'' = 8(x-1) \cdot (x-4) + 4(x-1)^2$$

$$= 4(x-1)(2x-8+x-1) = 4(x-1)(3x-9)$$

$$= 12(x-1)(x-3)$$

$$y'' = 0 \Rightarrow x=1 \text{ or } x=3$$



y is \cup for $x > 3$ or $x < 1$; y is \cap for $1 < x < 3$

(b) [5 points] Determine all inflection points (x, y) for the graph of $f(x) = \sin x - \cos x$ on the interval $[0, 2\pi]$. DO NOT GRAPH THE FUNCTION.

$$f'(x) = \cos x + \sin x$$

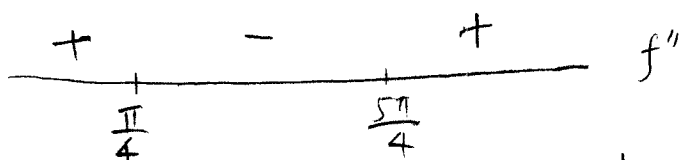
$$f''(x) = -\sin x + \cos x$$

$$f''(x) = 0 \Rightarrow -\sin x + \cos x = 0$$

$$\Rightarrow \sin x = \cos x \Rightarrow \frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4} \text{ or } x = \frac{5\pi}{4}$$



inflection points: $x = \frac{1}{4}\pi, x = \frac{5\pi}{4}$

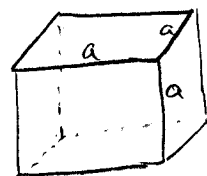
Problem 3

[10 points]

(a) [5 points] The volume of a cube is increasing at a rate of $3 \text{ in}^3/\text{sec}$. How fast is the surface area changing when each edge is 2 inches long?

$$V = a^3$$

$$S = 6a^2$$



$$\frac{dV}{dt} = 3 \text{ in}^3/\text{sec}$$

$$\Rightarrow 3 = 3a^2 \frac{da}{dt} \Rightarrow \frac{da}{dt} = \frac{1}{a^2}$$

$$\frac{dS}{dt} = 12a \cdot \frac{da}{dt}$$

$$\text{So at } a = 2 \text{ inch, } \frac{dS}{dt} = 12 \cdot 2 \cdot \frac{1}{2^2} = 6 \text{ in}^2/\text{sec}$$

(b) [5 points] Assuming that y is a function of x and that $y^3 - xy = 4$. Find an equation of the line perpendicular to the graph of this equation at $x = 0$.

Taking implicit derivative:

$$\frac{d}{dx} [y^3 - xy] = \frac{d}{dx} [4] \Rightarrow 3y^2 \frac{dy}{dx} - y - x \frac{dy}{dx} = 0$$

$$\Rightarrow (3y^2 - x) \frac{dy}{dx} = y \Rightarrow \frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$$\text{At } x = 0, y \text{ satisfies: } y^3 - 0 \cdot y = 4 \Rightarrow y = \sqrt[3]{4}$$

$$\text{So the slope of the tangent line of the function is: } m = \frac{\sqrt[3]{4}}{3(\sqrt[3]{4})^2 - 0} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{4}}$$

slope of the perpendicular line is:

$$m_1 = -\frac{1}{m} = -3 \cdot \sqrt[3]{4}$$

equation of the line is

$$y - \sqrt[3]{4} = -3 \cdot \sqrt[3]{4} (x - 0) \Rightarrow y = \sqrt[3]{4} + 3 \cdot \sqrt[3]{4} x$$

Problem 4

[10 points]

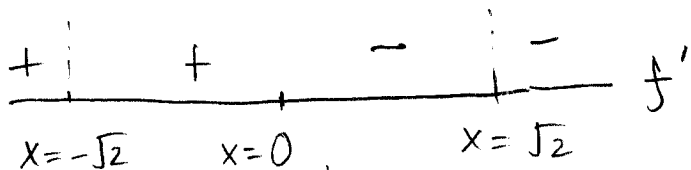
For the following function f , state the domain and determine all absolute and relative minimum and maximum values, inflection points, and x - and y -intercepts. State clearly the x -values for which f is increasing (\uparrow), decreasing (\downarrow), concave up (\cup), concave down (\cap). Find equations for all vertical and horizontal asymptotes. Sketch the graph of f carefully.

$$f(x) = \frac{x^2 + 1}{x^2 - 2}$$

The domain is: $x \neq \sqrt{2}, x \neq -\sqrt{2}$

$$f'(x) = \frac{2x(x^2 - 2) - (x^2 + 1)(2x)}{(x^2 - 2)^2} = \frac{-6x}{(x^2 - 2)^2}$$

$$f'(x) = 0 \Rightarrow x = 0$$



$f(x)$ is \uparrow when $x < 0$ ($x \neq -\sqrt{2}$)

$f(x)$ is \downarrow when $x > 0$ ($x \neq \sqrt{2}$)

$Y = -\frac{1}{2}$
rel. maximum

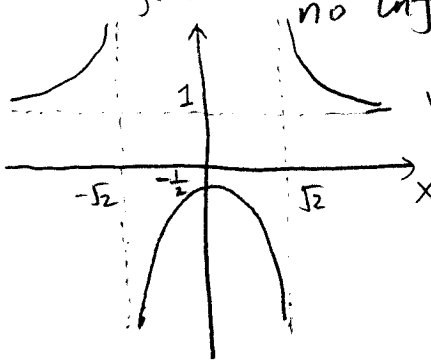
$$f''(x) = \frac{-6 \cdot (x^2 - 2)^2 - (-6x) \cdot 2(x^2 - 2) \cdot 2x}{(x^2 - 2)^4}$$

$$= \frac{(x^2 - 2)(-6x^2 + 12 + 24x^2)}{(x^2 - 2)^4} = \frac{18x^2 + 12}{(x^2 - 2)^3} = \frac{6(3x^2 + 2)}{(x^2 - 2)^3}$$

$f''(x) = 0 \Rightarrow 6 \cdot (3x^2 + 2) = 0 \Rightarrow$ no solution
 $f(x)$ is \cup for $x > \sqrt{2}$ or $x < -\sqrt{2}$
 $f(x)$ is \cap for $-\sqrt{2} < x < \sqrt{2}$

no inflection point

Y -intercepts: $x = 0, Y = -\frac{1}{2}$
 X -intercepts: none



vertical asymptotes

$$x = \pm\sqrt{2}$$

horizontal asymp

$$y = 1$$

$$\left\{ \begin{aligned} \lim_{x \rightarrow \sqrt{2}^+} \frac{x^2 + 1}{x^2 - 2} &= +\infty, & \lim_{x \rightarrow \sqrt{2}^-} \frac{x^2 + 1}{x^2 - 2} &= -\infty \\ \lim_{x \rightarrow \sqrt{2}^+} \frac{x^2 + 1}{x^2 - 2} &= -\infty, & \lim_{x \rightarrow -\sqrt{2}^-} \frac{x^2 + 1}{x^2 - 2} &= +\infty \\ \lim_{x \rightarrow \pm\infty} \frac{x^2 + 1}{x^2 - 2} &= \lim_{x \rightarrow \pm\infty} \frac{1 + \frac{1}{x^2}}{1 - \frac{2}{x^2}} = 1 \end{aligned} \right.$$

Problem 5 (Multiple-Choice and True-and-False Questions)

[10 points]

(1) The second derivative of $f(x) = x + 100$ is:

- (a) $f''(x) = 0$ (b) $f''(x) = 1$ (c) $f''(x) = 100$ (d) $f''(x)$ is undefined

(2) The domain of $f(x) = 1/(x^2 - 1)$ is

- (a) all real numbers (b) all real numbers except for $x = 1$ or $x = -1$
 (c) $-1 \leq x \leq 1$ (d) $-1 < x < 1$

(3) If

$$f'(x) = 2 - \frac{2}{x^{1/3}} = \frac{2(x^{1/3} - 1)}{x^{1/3}} = 0,$$

we must have

- (a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) no solution

(4) The second derivative represents the rate of change of the first derivative.

- True False

(5) If f has a relative minima when $x = c$, then it must be that either $f'(c) = 0$ or $f'(c)$ is undefined.

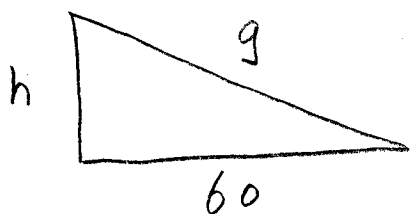
- True False

Bonus Problem

[10 points]

The problem below is optional, and worths 10 points.

A watermelon is dropped from a height of 224 feet. It will strike the ground 60 feet from where you are standing. How fast is the distance between you and the watermelon changing after it has fallen for 3 seconds?



$$h = -16t^2 + 224$$

$$g^2 = h^2 + 60^2$$

$$\frac{dh}{dt} = -32t$$

$$2g \frac{dg}{dt} = 2h \cdot \frac{dh}{dt} + 0$$

$$\Rightarrow \frac{dg}{dt} = \frac{h \cdot \frac{dh}{dt}}{g}$$

$$\text{At } t = 3 \text{ sec, } h = -16 \cdot 9 + 224 = 80 \text{ feet}$$

$$g = \sqrt{60^2 + 80^2} = 100 \text{ feet}$$

$$\frac{dg}{dt} = \frac{80 \cdot (-32 \times 3)}{100} = -76.8 \text{ feet/sec}$$