Problem 1 [10 points]

Let f(x) be given by: $f(x) = \begin{cases} 2x - 4\tan(x) & \text{if } x \leq 0 \\ x^a + x^2 & \text{if } 0 < x < 1 \\ 9x - 7 & \text{if } x \geq 1 \end{cases}$ where a is a real number.

(a) [5 points] Is the function continuous at $x = 0$? Why or why not?

at $x = 0$, $f(x) = 2 \cdot 0 + 4 \cdot \tan(0) = 0$

$\lim_{x \to 0^+} x^a + x^2 = 0^a + 0^2 = 0$ (if $a > 0$), so

$f(x)$ is continuous if $a > 0$.

when $a < 0$, $\lim_{x \to 0^+} x^a + x^2 = \lim_{x \to 0^+} \frac{1}{x^{|a|}} + x^2$

is undefined

Therefore, $f(x)$ is not continuous when $a < 0$

(b) [5 points] For what values of a (if one exists) is f(x) differentiable at x= 1? (Hint: power rule) For each of those values of a, find $f'(1)$.

At $x = 1$

$x^a + x^2 = 1^a + 1^2 = 1 + 1 = 2$

$9x - 7 = 9 \cdot 1 - 7 = 2$

So $f(x)$ is continuous

$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} (x^a + x^2)' = \lim_{x \to 1^-} a \cdot x^{a-1} + 2x = a \cdot 1^{a-1} + 2 \cdot 1 = 2 + a$

$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} f'(9x - 7)' = 9 - 7 = 2$

$2 + a = 2 \Rightarrow a = 0.$ and $f'(1) = 2$

Problem 2 [10 points]

(a) [5 points] Solve the following trigonometry equation for $0 \leq \theta \leq 2\pi$:

$$2\cos^2\theta - \cos\theta = 1$$

Let $x = \cos\theta$, $\quad 2x^2 - x = 1$

$\Rightarrow \quad 2x^2 - x - 1 = 0 \Rightarrow (2x+1)(x-1) = 0$

$\Rightarrow \quad x = -\frac{1}{2} \quad$ or $\quad x = 1$

when $\cos\theta = -\frac{1}{2}$, $\quad \theta = \frac{2\pi}{3}$, or $\frac{4\pi}{3}$

when $\cos\theta = 1$, $\quad \theta = 0$, or 2π

(b) [5 points] Use the limit definition of derivative to compute the derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} 2x + h = 2x$$

Problem 3 [20 points]

Find the derivative $\frac{dy}{dx}$ for each of the following equation. (a) [5 points] $y = \frac{1}{x^2+1}$

$$y' = \left((x^2+1)^{-1} \right)' = (-1) \cdot (x^2+1) \cdot 2x$$

$$= -2x(x^2+1)$$

(b) [5 points] $y = \sin(2x^2)$

$$y' = \cos(2x^2) \cdot 4x$$

$$= 4x \cos(2x^2)$$

(c) [5 points] $x^2 + y^2 = 4$

$$\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}\left[4\right]$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

(d) [5 points]

$$\frac{xy - y^2}{y - x} = 4$$

$$\Rightarrow \frac{y(x-y)}{y-x} = 4$$

$$\Rightarrow y = 4 \qquad (x \neq y)$$

$$\Rightarrow \frac{dy}{dx} = 0$$

Problem 4 [18 points]

(a) [6 points] Find the differential dy for the following function.

$$y = -(x-1)^3(x-5)$$

$$dy = \left[-(x-1)^3(x-5)\right]' \cdot dx$$

$$= \left(-3(x-1)^2 \cdot 1 \cdot (x-5) - (x-1)^3 \cdot 1\right) dx$$

$$= -(x-1)^2\left(3x-15+x-1\right) dx$$

$$= -(x-1)^2(4x-16)\,dx$$

$$= -4(x-1)^2(x-4)\,dx$$

(b) [6 points] Find the vertical and horizontal asymptotes of

$$f = \frac{x^2-2x-24}{x^2+x-12} = \frac{\cancel{(x+4)}(x-6)}{(x-3)\cancel{(x+4)}}$$

vertical asymptotes:

$$= \frac{x-6}{x-3} \quad (x \neq -4)$$

$$\lim_{x\to 3^+} \frac{x-6}{x-3} = \frac{3-6}{0^+} = -\infty$$

$$\lim_{x\to 3^-} \frac{x-6}{x-3} = \frac{3-6}{0^-} = +\infty$$

horizontal asymptotes.

$$\lim_{x\to\infty} \frac{x-6}{x-3} = \lim_{x\to\infty} \frac{\frac{x}{x}-\frac{6}{x}}{\frac{x}{x}-\frac{3}{x}} = \lim_{x\to\infty} \frac{1-\frac{6}{x}}{1-\frac{3}{x}} = 1$$

$$\lim_{x\to -\infty} \frac{x-6}{x-3} = \lim_{x\to -\infty} \frac{1-6/x}{1-3/x} = 1$$

(c) [6 points] Find the second derivative of $y = x^2\cos(3x)$

$$y' = 2x \cdot \cos 3x + x^2 \cdot (-\sin(3x) \cdot 3)$$

$$= 2x \cdot \cos 3x - 3x^2 \sin(3x)$$

$$y'' = 2 \cdot \cos 3x + 2x(-\sin(3x) \cdot 3) - 6x \cdot \sin 3x - 3x^2 \cdot \cos(3x) \cdot 3$$

$$= -12x\sin(3x) + 2\cos(3x) - 9x^2\cos(3x)$$

Problem 5 [12 points]

(a) [6 points] The surface area of a cube is increasing at a rate of 4 in^2/sec. How fast is the volume changing when the length of each edge is 2 inches long?

(b) [6 points] Find the equation for the line that pass through the point (0, 0) and is parallel to the tangent of $f(x) = x^2$ at $x = 1/2$.

Problem 6 [10 points]

The profit derived from selling x units of a particular product is modeled by the formula

$$P = 16x^2 + 30x - 170.$$

(a) [3 points] Find the formula for the deferential dP.

$$dP = (32x + 30)\,dx$$

(b) [3 points] What is the actual gain in profit obtained by increasing the sales from 20 to 21 units?

$$\Delta P = \left(16(21)^2 + 30 \cdot 21 - 170\right) - \left(16 \cdot (20)^2 + 30 \cdot 20 - 170\right)$$

$$= 16(21^2 - 20^2) + 30 \cdot (21 - 20)$$

$$= 16(21+20) \cdot (21-20) + 30$$

$$= 16 \times 41 + 30 = 656 + 30 = 686$$

(c) [4 points] Use the deferential dP to approximate the change in profit when the sales increase from 20 to 21 units.

$$dP = (32 \cdot 20 + 30) \cdot 1$$

$$= 640 + 30 = 670$$

Problem 7 (Multiple-Choices/True-and-False Questions) [20 points]

Please circle one and only one answer for problems 1-5.

(1)

$$\lim_{x \to \pi^-} \frac{\cos x}{x} =$$

(a) $-\pi$ (b) $1/\pi$ (c) ∞ (d) Do not exist (e) None of the above

(2) The domain of $f(x) = 1/\sqrt{1-x^2}$ is
(a) all real numbers (b) all real numbers except for $x = 1$ or $x = -1$
(c) $-1 \leq x \leq 1$ (d) $-1 < x < 1$

(3) If $y = \sin^2(4x)$ then $y'(\frac{\pi}{12}) =$

$y' = 2 \cdot \sin(4x) \cdot \cos(4x) \cdot 4$
$= 4 \cdot \sin(8x) = 8 \sin(4x) \cos(4x)$

(a) 8 (b) 0 (c) $4\sqrt{3}$ (d) $2\sqrt{2}$ (e) None of the above

(4)

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} =$$

(a) ∞ (b) 2 (c) $-\infty$ (d) -2 (e) None of the above

(5) The curve $y = x^2 + 10x$ has a horizontal tangent line when x is
(a) 5 (b) -5 (c) 10 (d) -10 (e) None of the above

$y' = 2x + 10$

Please read carefully each statement in Problems 6-10, and determine whether each is True or False.

(6) The product rule is $(f \cdot g)' = f' \cdot g + f \cdot g'$.
■ True ☐ False

(7) If $f(x) = x^2 + 1$ and $g(x) = \sqrt{(x)}$. Then $f(g(x)) = x + 1$.
■ True ☐ False

(8) If $(c, f(c))$ is a point of inflection of the graph of f, then the derivative of f must be zero.
☐ True ■ False

(9) $y = x^3$ has an inflection point at $(0, 0)$.
■ True ☐ False

$y' = 3x^2$
$y'' = 6x$

(10) The second derivative of $f(x) = \cos(x)$ is $f''(x) = \tan(x)$.
☐ True ■ False