16A Xiao Final Exam
June 10, 2009

Your Name:

Last 4 Digits of Your Student ID:

1.) It is a violation of the university honor code to, in any way, assist another person in the completion of this exam, copy answers from another student's exam, have another student take your exam for you. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
2.) No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.
3.) Using only a calculator to determine the value of limits will receive little credit.
4.) You will be graded on proper use of the derivative notation.
5.) Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam.
6.) Include units on answers where units are appropriate.
7.) Make sure you have 10 pages including the cover page.

Problem	1	2	3	4	5	6	7	Bonus	Total
Points	10	10	20	18	12	10	20	20	100 + 20
Score									

Problem 1 [10 points]

Find the limits below.

(a) [5 points]

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} =$$

$$\lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

(b) [5 points]

$$\lim_{x\to 0} \frac{x^2}{x} =$$

$$= \lim_{x\to 0} x = 0$$

Problem 2 [10 points]

(a) [5 points] Solve the following trigonometry equation for $0 \leq \theta \leq 2\pi$:

$$2\cos^2\theta - \cos\theta = 1$$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\Rightarrow \cos\theta = -\frac{1}{2} \quad \text{or} \quad \cos\theta = 1$$

$$\theta = \pi - \frac{\pi}{3}, \quad \pi + \frac{\pi}{3}, \quad 0, \quad 2\pi$$

(b) [5 points] Use the limit definition of derivative to compute the derivative of $f(x) = x^2$.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x$$

Problem 3 [20 points]

Find the derivative $\frac{dy}{dx}$ for each of the following equation.

(a) [5 points] $y = \frac{1}{x^2+1}$

$$y' = \frac{1' \cdot (X^2+1) - 1 \cdot (X^2+1)'}{(X^2+1)^2}$$

$$= \frac{-2X}{(X^2+1)^2}$$

(b) [5 points] $y = \sin(2x^2)$

$$y' = \left(\cos(2X^2)\right)(2X^2)'$$

$$= 4X\cos(2X^2)$$

(c) [5 points] $x^2 + y^2 = 4$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

(d) [5 points]

$$\frac{xy - y^2}{y - x} = 4$$

$$\Leftrightarrow \quad xy - y^2 = 4y - 4x$$

Take the derivative with respect to x

$$y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 4\frac{dy}{dx} - 4$$

$$(x - 2y - 4)\frac{dy}{dx} = -y - 4$$

$$\frac{dy}{dx} = \frac{-y-4}{x-2y-4}$$

Problem 4 [18 points]

(a) [6 points] Find the differential dy for the following function.

$$y = -(x-1)^3(x-5)$$

$$y' = [-(x-1)^3]'(x-5) - (x-1)^3(x-5)'$$

$$= -3(x-1)^2(x-5) - (x-1)^3$$

(b) [6 points] Find the vertical and horizontal asymptotes of

$$f = \frac{x^2 - 2x - 24}{x^2 + x - 12}$$

$$\lim_{x \to \infty} \frac{x^2-2x-24}{x^2+x-12} = 1$$ horizontal asymptote $y=1$

$$f = \frac{x^2-2x-24}{(x+4)(x-3)}$$

vertical asymptote $x=-4$ and $x=3$

$$\lim_{x \to -4} f(x) = \infty \qquad \lim_{x \to 3} f(x) = \infty.$$

(c) [6 points] Find the second derivative of $y = x^2 \cos(3x)$

$$y' = (x^2)' \cos(3x) + x^2 [\cos(3x)]'$$

$$= 2x\cos(3x) - 3x^2 \sin(3x)$$

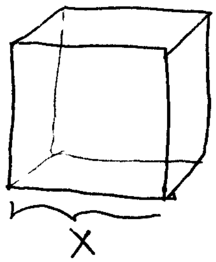
$$y'' = [(2x)' \cos 3x + 2x(\cos 3x)'] - [(3x^2)' \sin 3x + 3x^2(\sin 3x)']$$

$$= [2\cos 3x - 6x \sin 3x] - [6x \sin 3x + 9x^2 \cos 3x]$$

$$= 2\cos 3x - 12x\sin 3x - 9x^2 \cos 3x$$

Problem 5 [12 points]

(a) [6 points] The surface area of a cube is increasing at a rate of 4 in^2/sec. How fast is the volume changing when the length of each edge is 2 inches long?



surface area $S = 6X^2$ $\quad \frac{dS}{dt} = 4$

$$\frac{dS}{dt} = 12X\frac{dX}{dt} \Rightarrow X\frac{dX}{dt} = \frac{4}{12} = \frac{1}{3}$$

Volume $V = X^3$

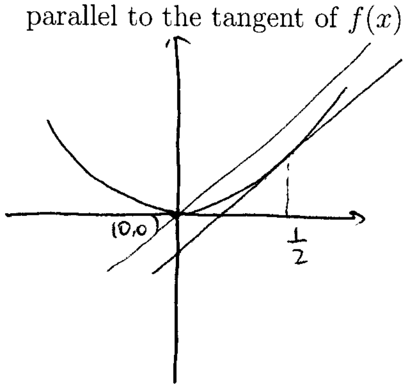
$X = 2$

$X\frac{dX}{dt} = \frac{1}{3}$

$$\frac{dV}{dt} = 3X^2\frac{dX}{dt} = 3X\left(X\frac{dX}{dt}\right)$$

$$= 3 \cdot 2 \cdot \left(\frac{1}{3}\right) = 2.$$

(b) [6 points] Find the equation for the line that pass through the point (0, 0) and is parallel to the tangent of $f(x) = x^2$ at $x = 1/2$.



tangent line at $X = 1/2$

m is the slope of the tangent line.

$y = x^2 \quad y' = 2x$

$$m = y'\big|_{x=1/2} = 2 \cdot \frac{1}{2} = 1$$

$$y - 0 = m(x - 0)$$

$$\Rightarrow y = x$$

Problem 6 [10 points]

The profit derived from selling x units of a particular product is modeled by the formula

$$P = 16x^2 + 30x - 170.$$

(a) [4 points] Find the formula for the deferential dP.

$$dP = (16X^2 + 30X - 170)' \, dX$$

$$dP = (32X + 30) \, dX$$

(b) [3 points] What is the actual gain in profit obtained by increasing the sales from 20 to 21 units?

$$P(20) = 16 \cdot 20^2 + 30 \cdot 20 - 170 = 6830$$

$$P(21) = 16 \cdot 21^2 + 30 \cdot 21 - 170 = 7516$$

$$P(21) - P(20) = 686$$

(c) [3 points] Use the deferential dP to approximate the change in profit when the sales increase from 20 to 21 units.

$$dP = (32X + 30)\Big|_{X=20} \, dX \qquad dX = 21 - 20 = 1$$

$$= (32 \cdot 20 + 30) \cdot 1$$

$$= 670$$

Problem 7 (Multiple-Choices/True-and-False Questions) [20 points]

Please circle one and only one answer for problems 1-5.

(1)

$$\lim_{x \to \pi^-} \frac{\cos x}{x} = \quad (1)\ \frac{\cos \pi}{\pi} = -\frac{1}{\pi}$$

(a) $-\pi$ (b) $1/\pi$ (c) ∞ (d) Do not exist (e) None of the above

(2) The domain of $f(x) = 1/\sqrt{1-x^2}$ is → (2) $1 - x^2 > 0. \Rightarrow -1 < x < 1$

(a) all real numbers (b) all real numbers except for $x = 1$ or $x = -1$

(c) $-1 \le x \le 1$ (d) $-1 < x < 1$

(3) If $y = \sin^2(4x)$ then $y'(\frac{\pi}{12}) =$

(a) 8 (b) 0 (c) $4\sqrt{3}$ (d) $2\sqrt{2}$ (e) None of the above

(3) $y' = 2\sin(4x) \cdot 4$
$= 8\sin 4x$

(4)

$$\lim_{x \to -3} \frac{x^2-9}{x+3} = \cancel{\lim_{x \to -3} (x+3)(x-3)}$$

(a) ∞ (b) 2 (c) $-\infty$ (d) -2 (e) None of the above

(4) $= \lim_{x \to -3} \frac{(x+3)(x-3)}{x+3}$
$= \lim_{x \to -3} (x-3) = -6$

(5) The curve $y = x^2 + 10x$ has a horizontal tangent line when x is

(a) 5 (b) -5 (c) 10 (d) -10 (e) None of the above

Please read carefully each statement in Problems 6-10, and determine whether each is True or False.

(5) $y' = 2x + 10 = 0$
$x = -5$.

(6) The product rule is $(f \cdot g)' = f' \cdot g + f \cdot g'$.

☒ True ☐ False

(7) If $f(x) = x^2 + 1$ and $g(x) = \sqrt{(x)}$. Then $f(g(x)) = x + 1$.

☒ True ☐ False

(8) If $(c, f(c))$ is a point of inflection of the graph of f, then the derivative of f must be zero.

☐ True ☒ False

(8) second derivative is zero or does not exist.

(9) $y = x^3$ has an inflection point at $(0,0)$.

☒ True ☐ False

(10) The second derivative of $f(x) = \cos(x)$ is $f''(x) = \tan(x)$.

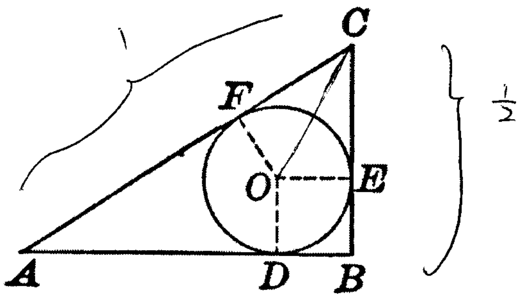
☐ True ☒ False

(10) $f'' = -\cos x$.

Bonus Problems [20 points]

The following problems are optional and are worth 10 points each.

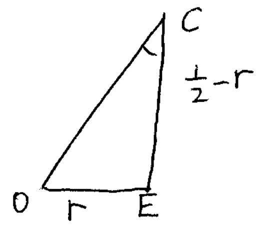
1.) A circle is inscribed in the right triangle shown below. We know that the length of the hypotenuse is $\overline{AC} = 1$, and the length of the shortest edge is $\overline{BC} = \frac{1}{2}$. What is the radius ($\overline{OE}, \overline{OF}$, or $\overline{OD}$) of the circle?



$\sin A = \frac{1}{2} \Rightarrow A = 30°$ $\quad C = 60°$

connect OC.

$\angle FCO = \angle OCE = 30°$.

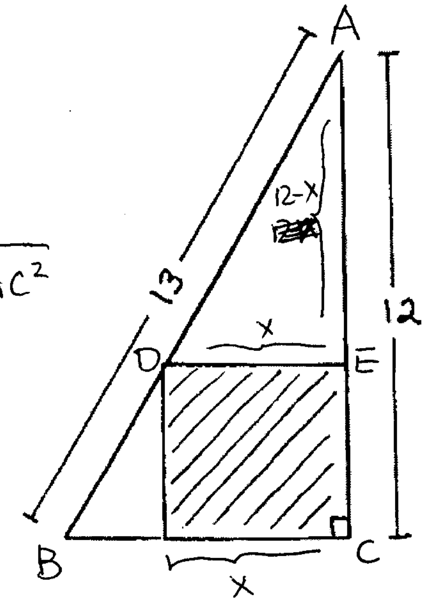


$\tan 30° = \dfrac{r}{\frac{1}{2} - r} = \dfrac{\sqrt{3}}{3}$

$3r = \dfrac{\sqrt{3}}{2} - \sqrt{3}r$

$(3r + \sqrt{3}r) = \dfrac{\sqrt{3}}{2}$ $\quad r = \dfrac{\sqrt{3}}{2(3+\sqrt{3})}$

2.) A square is inscribed in the given right triangle. Find the area of the square.



$\triangle ADE$ and $\triangle ABC$ are similar

$\dfrac{DE}{BC} = \dfrac{AE}{AC}$ $\quad BC = \sqrt{AB^2 - AC^2}$

$= 5$

$\dfrac{X}{5} = \dfrac{12 - X}{12}$

$12X = 60 - 5X$

$17X = 60$

$X = \dfrac{60}{17}$