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Approximate formulae for certain prolate spheroidal wave functions valid for large values of both order and band-limit

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Abstract

We construct asymptotic formulae for the approximation of certain prolate spheroidal wave functions and of the corresponding eigenvalues. We investigate two regimes: when the ratio c/m decays, and when both c and m grow, but the ratio c/m stays bounded. Both the regions of validity and the accuracies of the obtained expansions are illustrated with numerical examples.

Keywords: Prolate spheroidal wave functions; Band-limit; Asymptotic; Approximation

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1. Introduction

Originally, prolate spheroidal wave functions were discovered as the eigenvectors of the differential operator D_c defined by the formula

$$D_c(\psi)(x) = \left(1 - x^2\right)\psi''(x) - 2x\psi'(x) - c^2x^2\psi(x) \tag{1}$$

subject to the condition that ψ is continuous on the interval $[-1, 1]$. For each positive real c, there exists a countable set of real numbers $\chi_0 < \chi_1 < \chi_2 < \cdots$ for which the equation

$$\left(1 - x^2\right)\psi''(x) - 2x\psi'(x) + \left(\chi_j - c^2x^2\right)\psi(x) = 0 \tag{2}$$

has a continuous solution on $[-1, 1]$. Such coefficients are known as prolate eigenvalues, and the corresponding solutions of (2) are referred to as prolate spheroidal wave functions (PSWFs). About 45 years ago, it was observed (see [7,10,13]) that PSWFs are also eigenvectors of the integral operator

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$$F_c(\psi)(x) = \int_{-1}^{1} e^{icxt}\psi(t)\,\mathrm{d}t; \tag{3}$$

it is in this latter capacity that the PSWFs are of interest to the authors of the present paper, due to the obvious connections between (3) and Fourier transforms, Toeplitz matrices, antennas, Heisenberg principle, etc.

Despite their remarkable role as eigenvectors of (3), PSWFs have not been investigated in as much detail as several other classes of special functions; the reasons seem to be related to the fact that the classical scheme for their numerical evaluation—the so-called Bouwkamp method (see [1])—tends to fail for large values of c. In [14], we observe that a simple modification of the Bouwkamp scheme converts it into a reliable numerical tool for virtually all values of c that are likely to be encountered in practice; we also summarize a number of analytical properties of PSWFs, vaguely reminiscent of the properties of Bessel functions. In [15], we investigate the asymptotic behavior of ψ_m^c for large c and fixed m.

For any given c, PSWFs constitute an orthonormal basis on the interval $[-1, 1]$; numerical evidence is accumulating that in certain situations, they are preferable as a numerical tool to classical polynomial bases (such as Legendre and Chebychev polynomials). On the other hand, it has been observed that when PSWFs have to be dealt with analytically (i.e., when the time comes to put proofs behind facts observed numerically), the analysis tends to be much harder than that encountered while dealing with most classical special functions (orthogonal polynomials, Bessel functions, spherical harmonics, etc.). In the experience of the authors of the present paper, this relative difficulty has more to do with the paucity of published results about the PSWFs, rather than with the inherent difficulty of the subject. In this paper, we investigate PSWFs and the corresponding eigenvalues χ_m^c, λ_m^c in the regime when $m \geqslant c$; while most of the properties we derive are more or less obvious for $m \gg c$, the behavior of PSWFs is considerably more subtle when $m > c$, but m/c is not very large. The paper is meant to be a compendium of properties of PSWFs and the corresponding eigenvalues χ_m^c, λ_m^c that the authors found to be useful in their attempts to utilize PSWFs as a numerical tool. While most of the material presented here appears to be new, no serious effort has been made to separate original results from those published previously.

This paper contains two types of results. The first kind are expansions of various quantities (PSWFs, corresponding eigenvalues, etc.) into powers of c/m, valid when the ratio c/m is small (or, in some cases, not very large). Most of these expansions are of fairly high order (from 8 to 12) and have been obtained by the analysis of the three-term recursion connecting the coefficients of the prolate expansion of a function with the coefficients of the Legendre expansion of the same function (see Theorem 5 and Observation 6 below). Once the formulae in Section 4 are obtained, each of them is easily verified by substituting it into (2) and using the identities (7), (4), (13). This approach is very similar to that used in [15] to obtain high-frequency asymptotic formulae for PSWFs and their corresponding eigenvalues.

The second type of results found in this paper are asymptotic expansions of PSWFs and corresponding eigenvalues for large m and c, presented in Section 5. These expansions are of low order (from $1/\sqrt{(m \cdot c)}$ to $1/m^2$) and obtained via straightforward WKB analysis of Eq. (2). While the derivation of the formulae in Section 5 is straightforward and uses classical techniques (see, for example, [2]), it is quite detailed, and will be published at a later date; the results are included here for completeness.

The paper is organized as follows. In Section 2, we summarize a number of well-known mathematical facts to be used in this paper. In Section 3, we introduce the analytical apparatus used to derive the asymptotics presented in the paper. Section 4.1 is devoted to asymptotic formulae for eigenvalues and eigenvectors of (1), and in Section 4.2, we construct such formulae for eigenvalues of (3). Section 5 contains a different type of asymptotic formulae (see the preceding paragraph) and the numerical behavior of some of the presented approximations is illustrated in Section 6. Finally, Section 7 contains generalizations and conclusions.

2. Mathematical preliminaries

In this section, we introduce notation and summarize several well-known facts to be used in the rest of the paper.

2.1. Legendre polynomials

In agreement with standard practice, we will be denoting by P_n the classical Legendre polynomials, defined by the three-term recursion

$$P_{n+1}(x) = \frac{2 \cdot n + 1}{n+1} \cdot x \cdot P_n(x) - \frac{n}{n+1} \cdot P_{n-1}(x), \tag{4}$$

with the initial conditions

$$P_0(x) = 1, \qquad P_1(x) = x; \tag{5}$$

as is well known,

$$P_k(1) = 1 \tag{6}$$

for all $k = 0, 1, 2, \ldots$, and each of the polynomials P_k satisfies the differential equation

$$\left(1 - x^2\right) \cdot \frac{\mathrm{d}^2 P_k(x)}{\mathrm{d}x^2} + 2 \cdot x \cdot \frac{\mathrm{d}P_k(x)}{\mathrm{d}x} + k \cdot (k+1) \cdot P_k(x) = 0. \tag{7}$$

The following two lemmas summarize several well-known facts about Legendre polynomials. All of these facts can be found, for example, in [5].

Lemma 1. *For any positive integer m,*

$$\int_{-1}^{1} x^m P_m(x)\,\mathrm{d}x = \frac{\sqrt{\pi} m!}{2^m \Gamma(m + \frac{3}{2})} \tag{8}$$

and for any complex z,

$$P_m^{(m)}(z) = \frac{(2m)!}{2^m m!}. \tag{9}$$

Lemma 2. *Suppose that k, n are non-negative integers, $n \leqslant k$, and $k - n$ is even. Then*

$$\left| \int_{-1}^{1} x^k P_n(x)\,\mathrm{d}x \right| = \frac{\sqrt{\pi} k!}{2^k \left(\frac{k-m}{2}\right)! \Gamma(k + \frac{3}{2})}. \tag{10}$$

If $n > k$ or $k - n$ is odd,

$$\int_{-1}^{1} x^k P_n(x)\,\mathrm{d}x = 0. \tag{11}$$

The polynomials defined by the formulae (4), (5) are orthogonal on the interval $[-1, 1]$; however, they are not orthonormal, since for each $n \geqslant 0$,

$$\int_{-1}^{1} \left(P_n(x)\right)^2 \mathrm{d}x = \frac{1}{n + 1/2}; \tag{12}$$

the normalized version of the Legendre polynomials will be denoted by $\bar{P}_n$, so that

$$\bar{P}_n(x) = \frac{P_n(x)}{\sqrt{(n + 1/2)}}. \tag{13}$$

Thus, $\bar{P}_n$ constitute an orthonormal basis in $L^2[-1, 1]$.

2.2. Elliptic integrals

Incomplete elliptic integrals $F(x,a)$, $E(x,a)$ are defined by the formulae

$$F(x,a)=\int_0^x \frac{\mathrm{d}t}{\sqrt{(1-a^2\cdot\sin^2(t))}}, \tag{14}$$

$$E(x,a)=\int_0^x \sqrt{\big(1-a^2\cdot\sin^2(t)\big)}\,\mathrm{d}t, \tag{15}$$

respectively. Complete elliptic integrals $F(a)$, $E(a)$ are defined by the formulae

$$F(a)=F(\pi/2,a)=\int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{(1-a^2\cdot\sin^2(t))}}, \tag{16}$$

$$E(a)=E(\pi/2,a)=\int_0^{\pi/2} \sqrt{\big(1-a^2\cdot\sin^2(t)\big)}\,\mathrm{d}t. \tag{17}$$

We will denote by $\tilde{E}$ the function inverse to E, so that

$$\tilde{E}\big(E(a)\big)=a, \tag{18}$$

and by $\tilde{G}$ the function defined by the formula

$$\tilde{G}\big(E(x,a),a\big)=x; \tag{19}$$

in other words, for a fixed a, the function $\tilde{G}$ is the inverse of $E(x,a)$ with respect to the argument x.

Finally, we will denote by

$$f=\bar{E}(x) \tag{20}$$

the solution of the equation

$$\int_0^{\pi/2} \sqrt{\big(f-\sin^2(t)\big)}\,\mathrm{d}t=x \tag{21}$$

(viewed as an equation with respect to f) and observe the obvious connection between $\bar{E}$ and $\tilde{E}$.

2.3. Prolate spheroidal wave functions

In this section, we summarize a number of analytical properties of the prolate spheroidal wave functions. Unless stated otherwise, all facts collected below can be found in [7,13].

For any $c>0$, we denote by F_c the operator $L^2[-1,1]\to L^2[-1,1]$ defined by the formula

$$F_c(\varphi)(x)=\int_{-1}^{1} e^{icxt}\varphi(t)\,\mathrm{d}t. \tag{22}$$

In other words, F_c is a finite Fourier integral operator depending on c, a parameter we frequently refer to as the *band-limit* of F_c.

Clearly, F_c is compact; we denote by $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots,$ the eigenvalues of F_c in decreasing order such that $|\lambda_{n-1}| \geqslant |\lambda_n|$ for all natural n, and denote by ψ_n the corresponding eigenfunctions. In other words, for all non-negative integer n, λ_n and ψ_n satisfy the integral equation

$$\lambda_n \psi_n(x) = \int_{-1}^{1} e^{icxt} \psi_n(t)\,\mathrm{d}t \tag{23}$$

for all $x \in [-1, 1]$. In this paper, we adopt the convention that the functions are normalized such that $\|\psi_n\|_{[-1,1]} = 1$ for all n. The following theorem is a combination of several lemmas from [3,4,6,13].

Theorem 3. *For any positive real c, the eigenfunctions $\psi_0, \psi_1, \ldots,$ of the operator F_c are purely real, orthonormal and complete in $L^2[-1, 1]$. The even-numbered eigenfunctions are even and the odd-numbered ones are odd. All eigenvalues of F_c are non-zero and simple; the even-numbered eigenvalues are purely real and the odd-numbered ones are purely imaginary; in particular, $\lambda_n = i^n|\lambda_n|$. For each $n = 0, 1, 2, \ldots,$ the function ψ_n is an entire function of the two complex variables c, x.*

We define the self-adjoint operator $Q_c : L^2[-1, 1] \to L^2[-1, 1]$ by the formula

$$Q_c(\varphi) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sin(c \cdot (x - t))}{x - t} \varphi(t)\,\mathrm{d}t, \tag{24}$$

which relates to F_c in the form

$$Q_c = \frac{c}{2\pi} \cdot F_c^* \cdot F_c. \tag{25}$$

Consequently, Q_c has the same eigenfunctions as F_c, and the nth (in descending order) eigenvalue μ_n of Q_c is given by the formula

$$\mu_n = \frac{c}{2\pi} \cdot |\lambda_n|^2. \tag{26}$$

The eigenvalues λ are analytic functions of c and satisfy the differential equations

$$\frac{\partial \lambda_m}{\partial c} = \lambda_m \frac{2(\psi_m^c(1))^2 - 1}{2c}, \tag{27}$$

or, equivalently,

$$\frac{\partial \log(\lambda_m)}{\partial c} = \frac{1}{2c}\left(2\left(\psi_m^c(1)\right)^2 - 1\right) \tag{28}$$

(see [3,11]).

Obviously, the operator Q_c is closely related to the operator $P_c : L^2[-\infty, \infty] \to L^2[-\infty, \infty]$ defined by the formula

$$P_c(\varphi) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\sin(c \cdot (x - t))}{x - t} \cdot \varphi(t)\,\mathrm{d}t, \tag{29}$$

which, as is well known, is the orthogonal projection operator onto the space of functions of band limit c on $(-\infty, \infty)$.

For large c, the spectrum of Q_c consists of three parts: about $2c/\pi$ eigenvalues that are very close to 1, followed by order $\log(c)$ eigenvalues which decay exponentially from 1 to nearly 0; the remaining eigenvalues are all very close to zero. More detailed discussions of the structure of the spectrum of Q_c can be found in [8,14], and a number of other places.

By a remarkable coincidence, the eigenfunctions $\psi_0, \psi_1, \ldots, \psi_n$ of the operator Q_c turn out to be the prolate spheroidal wave functions, well known from classical mathematical physics (see, for example, [9]). The following theorem formalizes this statement. It is proved in a considerably more general form in [3,10].

Theorem 4. *For any $c > 0$, there exists a strictly increasing sequence of positive real numbers $\chi_0, \chi_1, \ldots,$ such that for each integer $n \geqslant 0$, the differential equation*

$$\big(1-x^2\big)\psi''(x) - 2x\psi'(x) + \big(\chi_n - c^2x^2\big)\psi(x) = 0 \tag{30}$$

has a solution that is continuous and bounded on the interval $[-1, 1]$. *Moreover, for each integer $n \geqslant 0$, the function ψ_n (defined in Theorem* 3) *is the solution of* (30).

3. Analytical apparatus

In this section, we build a number of analytical tools for the construction of the formulae in this paper.

3.1. Legendre coefficients and three-term recursions

Prolate spheroidal wave functions and Legendre polynomials are closely related, as can be observed from the similarities between their corresponding differential equations (30) and (7).

Since the scaled Legendre polynomials $\bar{P}_k$ constitute an orthonormal basis in $L^2[-1, 1]$, ψ_m^c can be expanded in the Legendre series

$$\psi_m^c(x) = \sum_{k=0}^{\infty} \beta_k^m \bar{P}_k(x), \tag{31}$$

with the coefficients β_k^m given by the formula

$$\beta_k^m = \int_{-1}^{1} \psi_m^c(x)\bar{P}_k(x)\,\mathrm{d}x; \tag{32}$$

since ψ_m^c is analytic on the interval $[-1, 1]$, the coefficients β_k^m decay exponentially once k is sufficiently large (a more detailed discussion of the rate of decay of $\{\beta_k^m\}$ can be found in [14]).

For small c, the connection between the functions ψ_m^c and Legendre polynomials $\bar{P}_m$ is well known. *Inter alia*,

$$\psi_m^c(x) = \bar{P}_m(x) + O\big(c^2\big) \tag{33}$$

for all non-negative integer m and $x \in [-1, 1]$ (see, for example, [11]).

Substituting (31) into (30) and using (7), (4), (13), we obtain the well-known three-term recursion

$$\begin{aligned} &\frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+5)(2k+1)}} \cdot c^2 \cdot \beta_{k+2}^m + \left(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2 - \chi_n\right) \cdot \beta_k^m \\ &\quad + \frac{k(k-1)}{(2k-1)\sqrt{(2k-3)(2k+1)}} \cdot c^2 \cdot \beta_{k-2}^m = 0. \end{aligned} \tag{34}$$

Introducing the notation

$$\beta^m = \big(\beta_0^m, \beta_1^m, \beta_2^m, \ldots\big) \in l^2 \tag{35}$$

for all $m = 0, 1, 2, \ldots,$ we restate (34) in a slightly different form in the following theorem.

Theorem 5. *Suppose that χ_m's are eigenvalues of the differential operator* (30) *and that β^m's are vectors defined in* (35). *Suppose further that a matrix A is given by the formulae*

$$A_{k,k} = k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2, \tag{36}$$

$$A_{k,k+2} = \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2, \tag{37}$$

$$A_{k+2,k} = \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2 \tag{38}$$

for all $k = 0, 1, 2, \ldots$, *with the remainder of the entries being zero. Then,* χ_m *are the eigenvalues of* A, *with* β^m *corresponding eigenvectors.*

In other words, in the basis consisting of the functions $\bar{P}_0, \bar{P}_1, \ldots, \bar{P}_k, \ldots$, the differential equation (30) has the form

$$(A - \chi_m \cdot I) \cdot \beta^m = 0. \tag{39}$$

Observation 6. The matrix A separates into two symmetric, tridiagonal matrices A_{even}, A_{odd}, with the former consisting of elements of A on the even-numbered rows and even-numbered columns and the latter elements on odd-numbered rows and odd-numbered columns. While these two matrices are of infinite dimensions and their entries do not decay much with increasing row or column numbers, the coordinates β_k^m of the eigenvectors β^m decay rapidly once k is sufficiently large.

3.2. Inverse power method as an analytical tool

In this section, we outline a scheme for the construction of asymptotic expansions in powers of c and $1/m$ for functions ψ_m^c and for certain quantities associated with them; the expansions constructed in this fashion are valid for large m (see Section 4.1). It must be pointed out that once expansions (64), (65), (67), (68) have been constructed, substituting them into (2) and using the identities (7), (4), (13) readily (though somewhat messily) proves their validity. Thus, this section should be viewed as a somewhat heuristic description of the procedure via which the expansions (64), (65), (67), (68) have been obtained; while the contents of this section can be made rigorous, the resulting proofs are long, elementary and add little to the subject of this paper.

We start with introducing the notation

$$a_k = \frac{2k(k+1) - 1}{(2k+3)(2k-1)} \cdot c^2, \tag{40}$$

$$b_k = \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2, \tag{41}$$

so that the doubly infinite symmetric tridiagonal matrices $A = (A_{i,j})$ assume the form

$$A_{k,k} = k(k+1) + a_k, \tag{42}$$

$$A_{k,k+2} = b_k, \tag{43}$$

$$A_{k+2,k} = b_k. \tag{44}$$

Denoting by $A^{\mu,\nu} = (A^{\mu,\nu})_{i,j}$ the square submatrix of either A_{even} or A_{odd} (see Observation 6) consisting of elements $A_{i,j}$ such that $\mu \leqslant i, j \leqslant \nu$ (see Fig. 1 for a depiction of $A^{(n-2),(n+2)}$), we start with the following observation.

Observation 7. For a fixed c and any $0 < \mu < \nu$, the matrix $A^{\mu,\nu}$ is diagonally dominant. Moreover, for sufficiently large μ

$$A_{k+1,k+1}^{\mu,\nu} - A_{k,k}^{\mu,\nu} > \left|A_{k+1,k}^{\mu,\nu}\right| + \left|A_{k,k+1}^{\mu,\nu}\right| \tag{45}$$

for all $\mu < k < \nu - 1$. Thus, for any sufficiently large μ,

$$\lambda_k\left(A^{\mu,\nu}\right) \sim 2k \cdot (2k+1), \tag{46}$$

with $\lambda_k(A^{\mu,\nu})$ denoting the kth eigenvalue of the matrix $A^{\mu,\nu}$.

Consider now the submatrix $A^{(n-k),(n+k)}$ of the matrix A_{even} (see Fig. 1, where the case of $k = 2$ is depicted). Obviously, for large n, it has an eigenvalue close to $\chi^0 = n(n+1)$, with the corresponding eigenvector close to

$$x^0 = (\overbrace{0, 0, \ldots, 0}^{k}, 1, \overbrace{0, \ldots, 0, 0}^{k}); \tag{47}$$

$(n-4)(n-3)+a_{n-4}$	b_{n-4}			
b_{n-4}	$(n-2)(n-1)+a_{n-2}$	b_n		
	b_n	$n(n+1)+a_n$	b_{n+2}	
		b_{n+2}	$(n+2)(n+3)+a_{n+2}$	b_{n+4}
			b_{n+4}	$(n+4)(n+5)+a_{n+4}$

Fig. 1. Matrix $A^{(n-2),(n+2)}$.

we would like to find an improved approximation to the eigenvalue of $A^{(n-k),(n+k)}$ closest to $n(n+1)$ and to the corresponding eigenvector. Employing the standard inverse power method with a shift, we form the matrix

$$B^{(n-k),(n+k)} = A^{(n-k),(n+k)} - \chi^0 \cdot I, \tag{48}$$

shown in Fig. 2 (with I denoting the identity) and evaluate the sequence of vectors

$$\left(B^{(n-k),(n+k)}\right)^{-1}\left(\chi^0\right), \qquad \left(B^{(n-k),(n+k)}\right)^{-2}\left(\chi^0\right), \qquad \left(B^{(n-k),(n+k)}\right)^{-3}\left(\chi^0\right), \qquad \ldots. \tag{49}$$

On inspection of the formulae (40)–(42), it is obvious that every coordinate of every element in the sequence (49) is a rational function of c, n. Now, constructing the sequence in the symbolic form (we used Mathematica), one can decompose it into a power series with respect to c, n; the results of such expansion are presented in (64), (65), (67), (68).

3.3. Connections between $\psi_m(1)$ and λ_m for large m

The principal purpose of this section is Theorem 9, providing an exact expression for the eigenvalues λ_m^c as functions of c, m and $\psi_m^c(1)$. Subsequently, asymptotic expansions for $\psi_m^c(1)$ are substituted into (58) provided by Theorem 9 to obtain asymptotic expansions for λ_m^c.

We start with a simple lemma describing the behavior of λ_m^c for small c.

Lemma 8. *For any $m = 0, 1, 2, \ldots,$*

$$\lim_{c\to 0} \lambda_m^c \cdot \frac{(m!)^2\Gamma(m+\frac{3}{2})}{i^m c^m \sqrt{\pi}(2m)!} = 1. \tag{50}$$

$(n-4)(n-3)$ $+a_{n-4}$ $-n(n+1)$	b_{n-4}			
b_{n-4}	$(n-2)(n-1)$ $+a_{n-2}$ $-n(n+1)$	b_n		
	b_n	a_n	b_{n+2}	
		b_{n+2}	$(n+2)(n+3)$ $+a_{n+2}$ $-n(n+1)$	b_{n+4}
			b_{n+4}	$(n+4)(n+5)$ $+a_{n+4}$ $-n(n+1)$

Fig. 2. Matrix $B^{(n-2),(n+2)}$.

Proof. Differentiating (23) m times with respect to x and evaluating the result at $x = 0$, we obtain

$$\lambda_m^c \psi_m^{(m)}(0) = (ic)^m \int_{-1}^{1} t^m \psi_m(t)\,\mathrm{d}t \tag{51}$$

or

$$\frac{\lambda_m^c\ \psi_m^{(m)}(0)}{i^m c^m} = \int_{-1}^{1} t^m \psi_m(t)\,\mathrm{d}t; \tag{52}$$

therefore,

$$\lim_{c\to 0} \frac{\lambda_m^c\ \psi_m^{(m)}(0)}{i^m c^m} = \lim_{c\to 0} \int_{-1}^{1} t^m \psi_m(t)\,\mathrm{d}t. \tag{53}$$

Now, using the combination of (33), (8), (9), (10), and (13), we rewrite (53) as

$$\lim_{c\to 0} \frac{\lambda_m^c (2m)!\sqrt{(m+1/2)}}{i^m c^m 2^m m!} = \lim_{c\to 0} \sqrt{(m+1/2)} \int_{-1}^{1} t^m P_m(t)\,\mathrm{d}t \tag{54}$$

or

$$\lim_{c\to 0} \frac{\lambda_m^c (2m)!}{i^m c^m 2^m m!} = \frac{\sqrt{(\pi)}m!}{2^m \Gamma(m+\frac{3}{2})} \tag{55}$$

or, finally,

$$\lim_{c\to 0} \frac{\lambda_m^c}{c^m} = \frac{\sqrt{(\pi)}(2m)!i^m}{2^m \Gamma(m+\frac{3}{2})(m!)^2}. \tag{56}$$

The following theorem is one of principal results of this paper. It provides an explicit expression for λ_m^c in terms of $\psi_m(1)$. □

Theorem 9. *For any positive real c and integer $m \geqslant 0$,*

$$\lambda_m^c = \frac{i^m \sqrt{\pi} c^m (m!)^2}{(2m)!\Gamma(m+3/2)} \cdot e^{F(c)}, \tag{57}$$

where $F(c)$ is given by the formula

$$F(c) = \int_0^c \left(\frac{2(\psi_m^\tau(1))^2 - 1}{2\tau} - \frac{m}{\tau} \right) \mathrm{d}\tau. \tag{58}$$

Proof. Suppose that c_0, c are two positive real numbers such that $0 < c_0 < c$. Integrating (28) from c_0 to c, we obtain

$$\log(\lambda_m^c) = \log(\lambda_m^{c_0}) + \int_{c_0}^c \frac{1}{2t} \left(2\left(\psi_m^t(1)\right)^2 - 1\right) \mathrm{d}t. \tag{59}$$

Due to (33),

$$\frac{1}{2t}\left(2\left(\psi_m^t(1)\right)^2 - 1\right) = \frac{m}{t} + p(t,m), \tag{60}$$

with p a smooth function of t (see (67) below for a more detailed analysis of p); substituting (60) into (59), we have

$$\log(\lambda_m^c) = \log(\lambda_m^{c_0}) + m \cdot \left(\log(c) - \log(c_0)\right) + \int_{c_0}^c p(t,m)\,\mathrm{d}t \tag{61}$$

exponentiating (61) and using (50), we obtain

$$\lambda_m^c = \lambda_m^{c_0} \cdot c^m \cdot c_0^{-m} \cdot e^{\int_{c_0}^c p(t,m)\,\mathrm{d}t}. \tag{62}$$

Remembering that p is a smooth function of t and using (50), we rewrite (62) as

$$\lambda_m^c = \lim_{c_0 \to 0} \left(\frac{\lambda_m^{c_0}}{c_0^m} \right) \cdot c^m \cdot e^{\int_0^c p(t,m)\,\mathrm{d}t} = \frac{i^m \sqrt{\pi} (m!)^2 c^m}{(2m)!\Gamma(m+3/2)} \cdot e^{\int_0^c p(t,m)\,\mathrm{d}t}, \tag{63}$$

which we combine with (60) to obtain (57) and (58). □

4. Formulae for PSWFs and eigenvalues χ_m, λ_m

The procedure described in Section 3.2 has been implemented in Mathematica and used to obtain asymptotic expansions for the eigenvalues χ_m^c and eigenfunctions ψ_m^c of the differential operator (1); the obtained asymptotic expansions were verified by substitution into (2), with the help of (7), (4), (13). The resulting expansions are listed in this section; several numerical examples illustrating their effectiveness can be found in Section 6.

4.1. Formulae for χ_m^c, ψ_m^c, $\psi_m^c(1)$, $\psi_m^c(0)$

Theorem 10. *For all real $c > 0$ and integer $m > 0$,*

$$\chi_m^c = m(m+1) + \frac{c^2}{2} + \frac{c^2(4+c^2)}{32m^2} - \frac{c^2(4+c^2)}{32m^3} + \frac{c^2(28+13c^2)}{128m^4} - \frac{c^2(20+11c^2)}{64m^5} + \frac{c^2(3904+3936c^2+160c^4+5c^6)}{8192m^6} - \frac{c^2(5824+8416c^2+480c^4+15c^6)}{8192m^7} + c^2 \cdot O\left(\frac{c^8}{m^8}\right). \tag{64}$$

Theorem 11. *For all real $c > 0$, integer $m > 0$, and $x \in [-1, 1]$,*

$$
\begin{aligned}
\psi_m^c(x) = {} & \Bigg(\frac{c^{14}}{1352914698240m^7}\Bigg) \cdot \bar{P}_{m-14}(x) + \Bigg(\frac{c^{12}}{671088640m^7} + \frac{c^{12}}{12079595520m^6}\Bigg) \cdot \bar{P}_{m-12}(x) \\
& + \Bigg(\frac{c^{10}(153600 - c^4)}{193273528320m^7} + \frac{5c^{10}}{50331648m^6} + \frac{c^{10}}{125829120m^5}\Bigg) \cdot \bar{P}_{m-10}(x) \\
& + \Bigg(\frac{c^8(262400 - 11c^4)}{2013265920m^7} + \frac{c^8(55680 - c^4)}{2013265920m^6} + \frac{c^8}{196608m^5} + \frac{c^8}{1572864m^4}\Bigg) \cdot \bar{P}_{m-8}(x) \\
& + \Bigg(\frac{c^6(417955840 + 983040c^2 - 46080c^4 + c^8)}{64424509440m^7} + \frac{c^6(99072 - 13c^4)}{50331648m^6} \cdot \\
& + \frac{c^6(15616 - c^4)}{25165824m^5} + \frac{3c^6}{16384m^4} + \frac{c^6}{24576m^3}\Bigg) \cdot \bar{P}_{m-6}(x) \\
& + \Bigg(\frac{c^4(19169280 + 589824c^2 + 8064c^4 + c^8)}{201326592m^7} + \frac{c^4(27721728 + 393216c^2 - 10240c^4 + c^8)}{805306368m^6} \\
& + \frac{c^4(1920 - c^4)}{131072m^5} + \frac{c^4(2880 - c^4)}{393216m^4} + \frac{c^4}{256m^3} + \frac{c^4}{512m^2}\Bigg) \cdot \bar{P}_{m-4}(x) \\
& + \Bigg(\frac{c^2(3397386240 + 2415919104c^2 + 21528576c^4 - 1769472c^6 - 43008c^8 - c^{12})}{38654705664m^7} \\
& + \frac{c^2(1523712 + 491520c^2 + 29568c^4 + c^8)}{25165824m^6} + \frac{c^2(540672 + 98304c^2 - 4992c^4 + c^8)}{12582912m^5} \\
& + \frac{c^2(512 - c^4)}{16384m^4} + \frac{c^2(256 - c^4)}{8192m^3} + \frac{c^2}{32m^2} + \frac{c^2}{16m}\Bigg) \cdot \bar{P}_{m-2}(x) \\
& + \Bigg(1 + \frac{31948800c^4 + 589824c^6 + 2688c^8 + c^{12}}{201326592m^7} - \frac{46301184c^4 + 589824c^6 - 1152c^8 + c^{12}}{603979776m^6} \\
& + \frac{3840c^4 - c^8}{131072m^5} - \frac{4352c^4 - c^8}{262144m^4} + \frac{c^4}{256m^3} - \frac{c^4}{256m^2}\Bigg) \cdot \bar{P}_m(x) \\
& + \Bigg(\frac{c^2(-94296342528 - 11475615744c^2 + 53968896c^4 + 1769472c^6 - 12288c^8 + c^{12})}{38654705664m^7} \\
& + \frac{c^2(28164096 + 1474560c^2 - 66432c^4 + 11c^8)}{25165824m^6} - \frac{c^2(6832128 + 98304c^2 - 17280c^4 + c^8)}{12582912m^5} \\
& - \frac{c^2(-4608 + 7c^4)}{16384m^4} + \frac{c^2(-1280 + c^4)}{8192m^3} + \frac{3c^2}{32m^2} - \frac{c^2}{16m}\Bigg) \cdot \bar{P}_{m+2}(x) \\
& + \Bigg(\frac{-(c^4(297271296 + 2359296c^2 - 81152c^4 + 5c^8))}{402653184m^7} \\
& + \frac{c^4(185008128 + 393216c^2 - 61440c^4 + c^8)}{805306368m^6} - \frac{7c^4(4224 - c^4)}{393216m^5} + \frac{c^4(9792 - c^4)}{393216m^4} \\
& - \frac{c^4}{128m^3} + \frac{c^4}{512m^2}\Bigg) \cdot \bar{P}_{m+4}(x) \\
& + \Bigg(\frac{-c^6(2053734400 + 983040c^2 - 184320c^4 + c^8)}{64424509440m^7} + \frac{c^6(367872 - 23c^4)}{50331648m^6} \\
& - \frac{c^6(40192 - c^4)}{25165824m^5} + \frac{5c^6}{16384m^4} - \frac{c^6}{24576m^3}\Bigg) \cdot \bar{P}_{m+6}(x) \\
& + \Bigg(\frac{c^8(-775680 + 17c^4)}{2013265920m^7} + \frac{c^8(119680 - c^4)}{2013265920m^6} - \frac{c^8}{131072m^5} + \frac{c^8}{1572864m^4}\Bigg) \cdot \bar{P}_{m+8}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{c^{10}(-291840+c^4)}{193273528320m^7}+\frac{7c^{10}}{50331648m^6}-\frac{c^{10}}{125829120m^5}\right)\cdot\bar{P}_{m+10}(x)\\
&+\left(\frac{-c^{12}}{503316480m^7}+\frac{c^{12}}{12079595520m^6}\right)\cdot\bar{P}_{m+12}(x)\\
&+\left(\frac{-c^{14}}{1352914698240m^7}\right)\cdot\bar{P}_{m+14}(x)+\sqrt{m}\cdot O\left(\frac{c^{16}}{m^8}\right).
\end{aligned}
\tag{65}
$$

The following two theorems provide asymptotic expansions for $\psi_m^c(1)$, $\psi_m^c(0)$. It should be observed that the expansions below differ from expansion (65), in that (65) is an expansion in powers of $\frac{c^2}{m}$, while the expansions (67), (68) below (as well as the expansion (64) above) are in powers of $\frac{c}{m}$. In numerical terms, it means that expansions (67), (68), (64) produce reasonable accuracy whenever $m \geqslant c$, while (65) is useless unless $c < m^2$, which tends to be a rather restrictive condition.

Theorem 12. *For any $c > 0$ and integer $m \geqslant 0$,*

$$
\left|\psi_m^c(1)\right| < \sqrt{m+\frac{1}{2}}
\tag{66}
$$

and

$$
\begin{aligned}
\psi_m^c(1) = \sqrt{\frac{1}{2}+m}\Biggl(1 &- \frac{4c^2+c^4}{4(1+2m)^4} - \frac{8c^2+5c^4}{(1+2m)^6} - \frac{3072c^2+6800c^4+496c^6+17c^8}{64(1+2m)^8}\\
&- \frac{512c^2+4512c^4+1078c^6+55c^8}{2(1+2m)^{10}}\\
&- \frac{327680c^2+11668480c^4+6912192c^6+615944c^8+8844c^{10}+107c^{12}}{256(1+2m)^{12}}\\
&- \frac{393216c^2+56369152c^4+77568128c^6+12546920c^8+453880c^{10}+7125c^{12}}{64(1+2m)^{14}}\\
&- \frac{469762048c^2+270135197696c^4+847030386688c^6+247641926912c^8}{16384(1+2m)^{16}}\\
&- \frac{16252261888c^{10}+380878144c^{12}+2115648c^{14}+12573c^{16}}{16384(1+2m)^{16}}\\
&- \frac{33554432c^2+77265371136c^4+548143177728c^6+287562215936c^8}{256(1+2m)^{18}}\\
&- \frac{31109328352c^{10}+1111543216c^{12}+14134828c^{14}+101755c^{16}}{256(1+2m)^{18}}\\
&- \frac{38654705664c^2+356175128297472c^4+5699666788220928c^6}{65536(1+2m)^{20}}\\
&- \frac{5342103064903680c^8+92193079609 6512c^{10}+49298957839616c^{12}}{65536(1+2m)^{20}}\\
&- \frac{1022214431488c^{14}+10219338512c^{16}+28971260c^{18}+100327c^{20}}{65536(1+2m)^{20}}\\
&+ O\left(\frac{1}{m^{22}}\right)\Biggr).
\end{aligned}
\tag{67}
$$

Theorem 13. *For any $c > 0$ and integer $m \geqslant 1$,*

$$\begin{aligned}
\psi_m^c(0) = \sqrt{\frac{2}{\pi}}\Bigg(1 - \frac{2c^2+1}{16m^2} + \frac{2c^2+1}{16m^3} + \frac{8c^4-140c^2-3}{512m^4} - \frac{103680c^4-1399680c^2+4601}{3317760m^5} \\
- \frac{83980800c^6-976276800c^4+7865326800c^2-17805089}{10749542400m^6} \\
+ \frac{67184640c^6-557072640c^4+3430412748c^2+86309}{2866544640m^7} \\
+ \frac{1}{12383472844800m^8}\big(20785248000c^8-1986313881600c^6 \\
+ 4275336556800c^4-24430778197776c^2-1978643839\big) \\
- \frac{1}{24766945689600m^9}\big(166281984000c^8-13181626368000c^6 \\
+ 13528331886240c^4-79476319733832c^2-996600013\big) \\
- \frac{1}{99067782758400m^{10}}\big(82763078400c^{10}-4876030224000c^8 \\
+ 244104975820800c^6-21627672187536c^4-1276961293959615c^2+1381481054\big) \\
+ \frac{1}{792542262067200m^{11}}\big(3310523136000c^{10}-155133532800000c^8+6730329331837440c^6 \\
+ 149489524096664544c^4-25501720589940204c^2+4407192151\big)\Bigg) \\
+ O\left(\frac{c^{12}}{m^{12}}\right).
\end{aligned} \tag{68}$$

4.2. *Formulae for λ_m^c*

Theorem 9 expresses λ_m^c via c, m and $\psi_m^c(1)$, and Theorem 12 provides an expansion of $\psi_m^c(1)$ into powers of $1/m$. Combining these two observations (and carrying out the elementary but voluminous manipulations), we readily obtain expressions for λ_m^c, given in Theorem 14, proved by substituting (67) into (58) and carrying out the integration.

Theorem 14. *Suppose that $c > 0$ is a real number and m is a sufficiently large integer. Then*

$$\left|\lambda_m^c\right| < \frac{c^m\sqrt{\pi}(m!)^2}{(2m)!\Gamma(\frac{3}{2}+m)} \tag{69}$$

and

$$\left|\lambda_m^c\right| = \frac{c^m\sqrt{\pi}(m!)^2}{(2m)!\Gamma(\frac{3}{2}+m)} \cdot e^q \tag{70}$$

with

$$\begin{aligned}
q = -\frac{8c^2+c^4}{16(1+2m)^3} - \frac{16c^2+5c^4}{4(1+2m)^5} \\
- \frac{12288c^2+53952c^4+8512c^6+315c^8}{96(1+2m)^9} - \frac{12288c^2+13536c^4+640c^6+15c^8}{512(1+2m)^7} \\
- \frac{327680c^2+5824000c^4+2290560c^6+150990c^8+1656c^{10}+15c^{12}}{512(1+2m)^{11}} \\
- \frac{196608c^2+14082048c^4+12891648c^6+1554795c^8+44088c^{10}+550c^{12}}{64(1+2m)^{13}} \\
- \frac{384978886656c^{10}+7344995840c^{12}+33146880c^{14}+154245c^{16}}{3932160(1+2m)^{15}}
\end{aligned}$$

$$-\frac{56371445760c^2+16203707842560c^4+33835125309440c^6+7399764503040c^8}{3932160(1+2m)^{15}}$$
$$+\,O\left(\frac{c^{20}}{m^{19}}\right). \tag{71}$$

Remark 15. It should be observed that the expansion (71) has a drawback as a tool for approximating λ_m^c for large values of c. Indeed, some of the terms in (71) are of the form c^{k+1}/m^k with $k = 3, 7, 11, 15$. In other words, in order to obtain a prescribed accuracy for large c, the values of m have to grow *faster* than the values of c. For example, in the case of (71), m must grow as $c^{20/19}$. While numerical examples in Section 6 show that the resulting approximations are acceptable for fairly large ratios of c/m and fairly large c, in certain situations such estimates are not sufficient. To some extent, we attempt to remedy this problem in the following section.

Remark 16. Looking at the expansions (64), (67), (68) (but not the expansions (65), (71)), one is tempted to think that each of them represents the first few terms of a convergent series. We conjecture that indeed, there exist expansions approximating χ_m, $\psi_m^c(1)$, $\psi_m^c(0)$ that are convergent for $c < m$, and for which (64), (67), (68) are the first 7, 20, and 11 terms, respectively. Such convergent expansions are being investigated.

5. Results following from the WKB analysis of Eq. (2)

While all expansions in Section 4 are in terms of powers of c/m, it is often desirable to have approximate formulae for various quantities associated with the PSWFs that are valid when (for example) m increases, but c stays proportional to m. In this section, we list several such estimates. Since the results presented here are not the principal purpose of this paper, they are listed without proofs. The proofs will be published at a later date.

Theorem 17. *Suppose that $c \geqslant 1$ and that $m \geqslant c$ is an integer. Then*

$$\left|\frac{\frac{\psi_m(1)}{\psi_m(0)}}{\sqrt{(\pi/2)}\cdot(\chi_m^c)^{1/4}}-1\right| = O\left(\frac{1}{m^2}\right). \tag{72}$$

Theorem 18. *Suppose that $c \geqslant 1$ and that $m \geqslant c$ is an integer. Then*

$$\left|\frac{\chi_m^c}{c^2\cdot\bar{E}(m\cdot\pi/(2\cdot c))}-1\right| = O\left(\frac{1}{\sqrt{(m\cdot c)}}\right) \tag{73}$$

with $\bar{E}$ defined in (20), (21).

Theorem 19. *Suppose that $c \geqslant 1$ and that $m \geqslant c$ is an integer. Then*

$$\left|\frac{|\lambda_m^c|}{p_0(c,m)}-1\right| < \frac{1}{\sqrt{c\cdot m}}, \tag{74}$$

where $p_0(c,m)$ is defined by the formula

$$p_0(c,m) = \frac{e^{-\sqrt{\chi_m^c}\cdot(F(\sqrt{(1-c^2/\chi_m^c)})-E(\sqrt{(1-c^2/\chi_m^c)}))}}{\sqrt{c}\cdot\sqrt{2\cdot\pi}} \tag{75}$$

and E, F are the elliptic integral defined in (16), (17).

The above theorem provides an estimate that is effective for arbitrarily small c/m and gets tighter as c increases. When c/m is reasonably close to 1, a tighter estimate is provided by Theorem 20; however, the estimate provided by Theorem 20 *deteriorates* as c/m decreases.

Theorem 20. *Suppose that $b \leqslant 1$ and that $c = m\cdot b$. Then*

$$\lambda_m^c = p_0(c,m)\cdot p_1(c,m)\cdot\left(1+O\left(\frac{1}{m}\right)\right), \tag{76}$$

where p_0 *is defined in* (75), $p_1(c,m)$ *is defined by the formula*

$$p_1(c,m) = e^{\frac{(c^2\cdot F(\sqrt{(1-c^2/\chi_m^c)})-\chi_m^c\cdot E(\sqrt{(1-c^2/\chi_m^c)}))}{2\cdot(\chi_m^c-c^2)\cdot\sqrt{(\chi_m^c)}}} \tag{77}$$

and E, F *are the elliptic integrals defined in* (16), (17).

Theorem 21. *Suppose that* $b \leqslant 1$ *and that* $c = m \cdot b$. *Then for each* $k = 1, 2, \ldots, m$,

$$\left|x_k - \sin\left(\tilde{G}\left(\frac{k\cdot\pi}{\sqrt{(\chi_m)}}, \frac{c}{\sqrt{(\chi_m)}}\right)\right)\right| = O\left(\frac{1}{m^2}\right), \tag{78}$$

where x_k *denotes the kth root of* ψ_m^c, *and* $\tilde{G}$ *is defined in* (19).

6. Numerical results

The approximate formulae of Sections 4 and 5 were tested numerically. The calculations were performed in FORTRAN (the LINUX version from Lahey), in either double or extended precision, as needed. Tables 1–8 illustrate the numerical behavior of some of the approximations listed in Sections 4 and 5. The first two columns in each of the tables contain the bandlimit c and the degree m of the functions for which the calculation was performed. The third column contains the value of the parameter being approximated (calculated numerically with sufficient accuracy via the algorithm described in [14]) and the fourth column contains the approximation being tested. Finally, the fifth column contains the relative error of the approximation. More specifically,

Table 1
Illustration to Theorem 10

c	m	χ_m^c	$\tilde{\chi}_m^c$	$\lvert\frac{\tilde{\chi}_m^c}{\chi_m^c}-1\rvert$
10	10	0.1630966527E+03	0.1630714644E+03	0.154E−03
10	20	0.4707790239E+03	0.4707789503E+03	0.156E−06
100	200	0.4527786640E+05	0.4527786600E+05	0.869E−08
200	200	0.6146952030E+05	0.6146804219E+05	0.240E−04
1000	1000	0.1532864705E+07	0.1532827516E+07	0.242E−04
1000	700	0.1061453125E+07	0.1059550913E+07	0.179E−02
2000	2000	0.6129524565E+07	0.6129375490E+07	0.243E−04
2000	1400	0.4244667882E+07	0.4237027811E+07	0.179E−02
2000	2800	0.9906881564E+07	0.9906876723E+07	0.488E−06
2000	4000	0.1803528048E+08	0.1803528034E+08	0.746E−08
3000	3000	0.1378997985E+08	0.1378964417E+08	0.243E−04

Table 2
Illustration to Theorem 12

c	m	$\psi_m^c(1)$	$\tilde{\psi}_m^c(1)$	$\lvert\frac{\tilde{\psi}_m^c(1)}{\psi_m^c(1)}-1\rvert$
10	10	0.31902788747723E+01	0.31905422086869E+01	0.825E−04
10	20	0.45234567755878E+01	0.45234567756167E+01	0.638E−11
100	200	0.14146047761502E+02	0.14146047761502E+02	0.186E−13
200	200	0.13924692587818E+02	0.13924695713143E+02	0.224E−06
1000	1000	0.31101114869861E+02	0.31101121823492E+02	0.223E−06
1000	700	0.23879885740948E+02	0.23942265326233E+02	0.261E−02
2000	2000	0.43977324022989E+02	0.43977333899706E+02	0.224E−06
2000	1400	0.33756823931676E+02	0.33845812629714E+02	0.263E−02
2000	2800	0.52700849165636E+02	0.52700849168898E+02	0.618E−10
2000	4000	0.63187511935751E+02	0.63187511935752E+02	0.122E−13
3000	3000	0.53858430607666E+02	0.53858442724682E+02	0.224E−06

Table 3
Illustration to Theorem 13

c	m	$\psi_m^c(0)$	$\tilde{\psi}_m^c(0)$	$\left\lvert\frac{\tilde{\psi}_m^c(0)}{\psi_m^c(0)}-1\right\rvert$
10	10	0.711447032E+00	0.711718985E+00	0.382E−03
10	20	0.774580524E+00	0.774580586E+00	0.803E−07
100	200	0.773752792E+00	0.773752758E+00	0.444E−07
200	200	0.705601123E+00	0.705505463E+00	0.135E−03
1000	1000	0.705244981E+00	0.705146313E+00	0.139E−03
1000	700	0.593604249E+00	0.593320631E+00	0.477E−03
2000	2000	0.705199748E+00	0.705100786E+00	0.140E−03
2000	1400	0.593118898E+00	0.593118898E+00	0.457E−03
2000	2800	0.749503083E+00	0.749500780E+00	0.307E−05
2000	4000	0.773642977E+00	0.773642940E+00	0.469E−07
3000	3000	0.705184635E+00	0.705085580E+00	0.140E−03

Table 4
Illustration to Theorem 14

c	m	λ_m^c	$\tilde{\lambda}_m^c$	$\left\lvert\frac{\tilde{\lambda}_m^c}{\lambda_m^c}-1\right\rvert$
10	10	0.74448729094E−002	0.74460264240E−002	0.154E−03
10	20	0.11487284026E−009	0.11487284032E−009	0.503E−09
100	200	0.13980106499E−094	0.13980106499E−094	0.345E−10
200	200	0.50013133845E−035	0.50014447244E−035	0.262E−04
1000	1000	0.21812532717E−172	0.21815376765E−172	0.130E−03
1000	700	0.12446479810E−021	0.15087142653E−021	0.175E+00
2000	2000	0.83881420007E−344	0.83903370115E−344	0.261E−03
2000	1400	0.21530614133E−042	0.31719477252E−042	0.321E+00
2000	2800	0.95872406658E−883	0.95872445211E−883	0.402E−06
2000	4000	0.27032262499E−939	0.27032262504E−939	0.206E−09
3000	3000	0.37241503575E−515	0.37256142385E−515	0.392E−03

Table 5
Illustration to Theorem 17

c	m	$\frac{\psi_m^c(1)}{\psi_m^c(0)}$	r_m^c	$\left\lvert\frac{\frac{\psi_m^c(1)}{\psi_m^c(0)}}{r_m^c}-1\right\rvert$
10	10	0.4484211371E+01	0.4478898292E+01	0.118E−02
10	20	0.5839879308E+01	0.5838001665E+01	0.321E−03
100	200	0.1828238670E+02	0.1828232508E+02	0.337E−05
200	200	0.1973451022E+02	0.1973444398E+02	0.335E−05
1000	1000	0.4409973230E+02	0.4409972636E+02	0.134E−06
1000	700	0.4022863000E+02	0.4022862080E+02	0.228E−06
2000	2000	0.6236151404E+02	0.6236151194E+02	0.337E−07
2000	1400	0.5688802967E+02	0.5688802642E+02	0.572E−07
2000	2800	0.7031438603E+02	0.7031438479E+02	0.177E−07
2000	4000	0.8167528665E+02	0.8167528596E+02	0.846E−08
3000	3000	0.7637493487E+02	0.7637493373E+02	0.149E−07

- the parameter $\tilde{\chi}_m^c$ in Table 1 is defined in (64) in Theorem 10;
- the parameter $\tilde{\psi}_m^c(1)$ in Table 2 is defined in (67) in Theorem 12;
- the parameter $\tilde{\psi}_m^c(0)$ in Table 3 is defined in (68) in Theorem 13;
- the parameter $\tilde{\lambda}_m^c$ in Table 4 is defined in (70) in Theorem 14;
- the parameter r_m^c in Table 5 is defined as $r_m^c = \left(\frac{\pi}{2}\right)^{1/2} \cdot (\chi_m^c)^{1/4}$;
- the function p_0 in Table 7 is defined in (75) in Theorem 19;
- the function q in Table 8 is defined as $q(c,m) = p_0(c,m) \cdot p_1(c,m)$, with the functions p_0, p_1 defined in (75) and (77), respectively.

Table 6
Illustration to Theorem 18

c	m	χ_m^c	$c^2 \cdot \bar{E}(\frac{k\pi}{2c})$	$\lvert \frac{\chi_m^c}{c^2 \cdot \bar{E}(\frac{k\pi}{2c})} - 1 \rvert$
10	10	0.16310E+03	0.15319E+03	0.607E−01
10	20	0.47078E+03	0.45078E+03	0.424E−01
10	40	0.16902E+04	0.16502E+04	0.236E−01
100	200	0.45278E+05	0.45078E+05	0.440E−02
200	200	0.61470E+05	0.61276E+05	0.314E−02
1000	1000	0.15329E+07	0.15319E+07	0.630E−03
1000	700	0.10615E+07	0.10609E+07	0.538E−03
2000	2000	0.61295E+07	0.61276E+07	0.315E−03
2000	1400	0.42447E+07	0.42435E+07	0.268E−03
2000	2800	0.99069E+07	0.99041E+07	0.280E−03
2000	4000	0.18035E+08	0.18031E+08	0.221E−03
3000	3000	0.13790E+08	0.13787E+08	0.210E−03
4000	4000	0.24514E+08	0.24510E+08	0.157E−03

Table 7
Illustration to Theorem 19

c	m	$\lvert\lambda_m^c\rvert$	$p_0(c,m)$	$\lvert \frac{\lvert\lambda_m^c\rvert}{p_0(c,m)} - 1 \rvert$
10	10	0.74449E−002	0.76927E−002	0.332E−01
10	20	0.11487E−009	0.11744E−009	0.223E−01
10	40	0.31613E−031	0.32045E−031	0.136E−01
100	200	0.13980E−094	0.14011E−094	0.224E−02
200	200	0.50013E−035	0.50097E−035	0.167E−02
1000	1000	0.21813E−172	0.21820E−172	0.334E−03
1000	700	0.12446E−021	0.12451E−021	0.384E−03
2000	2000	0.83881E−344	0.83895E−344	0.167E−03
2000	1400	0.21531E−042	0.21535E−042	0.192E−03
2000	2800	0.95872E−883	0.95886E−883	0.140E−03
2000	4000	0.18428E−1877	0.18430E−1877	0.112E−03
3000	3000	0.37242E−515	0.37246E−515	0.111E−03
4000	4000	0.17536E−686	0.17538E−686	0.837E−04
5000	5000	0.852857E−858	0.85291E−858	0.670E−04

Table 8
Illustration to Theorem 20

c	m	$\lvert\lambda_m^c\rvert$	$q(c,m)$	$\lvert \frac{\lvert\lambda_m^c\rvert}{q(c,m)} - 1 \rvert$
10	10	0.744487290E−002	0.7446665216E−002	0.240E−03
10	20	0.114872840E−009	0.1150119590E−009	0.121E−02
10	40	0.316133912E−031	0.3167374120E−031	0.190E−02
100	200	0.139801065E−094	0.1398175798E−094	0.118E−03
200	200	0.500131338E−035	0.5001357815E−035	0.888E−05
1000	1000	0.218125327E−172	0.2181257109E−172	0.175E−05
1000	700	0.124464798E−021	0.1244648034E−021	0.421E−07
2000	2000	0.838814200E−344	0.8388149368E−344	0.878E−06
2000	1400	0.215306141E−042	0.2153061458E−042	0.209E−07
2000	2800	0.958724066E−883	0.9587269587E−883	0.301E−05
2000	4000	0.184289302E−1877	0.1842903879E−1877	0.588E−05
3000	3000	0.372415035E−515	0.3724152537E−515	0.585E−06
4000	4000	0.175367045E−686	0.1753671228E−686	0.438E−06
5000	5000	0.852857277E−858	0.8528575773E−858	0.351E−06

The final table (Table 9) illustrates the performance of the approximation (76) in the regime where the authors have encountered the need for such an approximation most frequently—when attempting to determine the smallest order m such that

Table 9
Detailed test of approximation (76)

c	ε	m	$\lvert\lambda_m^c\rvert$	$\tilde{\lambda}_m^c$	$\lvert\frac{\lvert\tilde{\lambda}_m^c\rvert}{\lambda_m^c}-1\rvert$
1	10^{-3}	4	0.483326329607E−03	0.492530484151E−03	0.190E−01
10	10^{-3}	12	0.366170506515E−03	0.366323581101E−03	0.418E−03
100	10^{-3}	71	0.555757103325E−03	0.555758812962E−03	0.307E−05
1000	10^{-3}	645	0.551063152176E−03	0.551076469521E−03	0.241E−04
4000	10^{-3}	2554	0.902127729344E−03	0.902361517780E−03	0.259E−03
1	10^{-7}	7	0.367923946418E−07	0.374888691762E−07	0.189E−01
10	10^{-7}	17	0.498748186136E−07	0.499224850446E−07	0.955E−03
100	10^{-7}	80	0.589250586187E−07	0.589252708516E−07	0.360E−05
1000	10^{-7}	658	0.790480013599E−07	0.790480016487E−07	0.365E−08
4000	10^{-7}	2571	0.614234522684E−07	0.614234522718E−07	0.549E−10
1	10^{-16}	13	0.749523262575E−17	0.737966150834E−17	0.156E−01
10	10^{-16}	27	0.158061093604E−16	0.158315550369E−16	0.160E−02
100	10^{-16}	97	0.541556572679E−16	0.541564849822E−16	0.152E−04
1000	10^{-16}	685	0.498403756086E−16	0.498403767525E−16	0.229E−07
4000	10^{-16}	2603	0.760602666182E−16	0.760602666448E−16	0.349E−09
1	10^{-34}	23	0.170865143278E−35	0.172893823007E−35	0.118E−01
10	10^{-34}	43	0.667546209121E−35	0.668837697107E−35	0.193E−02
100	10^{-34}	125	0.616701801071E−34	0.616729018212E−34	0.441E−04
1000	10^{-34}	731	0.465794829486E−34	0.465794877399E−34	0.102E−06
4000	10^{-34}	2661	0.556352488456E−34	0.556352489414E−34	0.172E−08

$$\lvert\lambda_m^c\rvert \leqslant \varepsilon \tag{79}$$

with c a prescribed band-limit and ε a reasonably small real number (see [14]). Here, $\tilde{\lambda}_m^c$ is defined by (76).

The following observations can be made from the tables and from the more extensive numerical experiments performed by the authors.

1. Approximations obtained via the inverse power method of Section 3.2 tend to be surprisingly accurate; all of them display rapid convergence as $c/m \to 0$. With the exception of (70) in Theorem 14, they produce at least three digits at $m \sim c$, and better than two digits at $m \sim 0.7c$. The estimate (70) produces almost 4 digits at $m \sim c$, but breaks down quickly once $m < c$; generally, this estimate should be used with care (see Remark 15).
2. Approximations obtained via the WKB analysis of Eq. (2) are slowly convergent (which is to be expected from (72), (73), (74), (76)), but quite robust. All of them are accurate to a few percent at $m \sim c \sim 10$ and produce better than two-digit accuracy at $m \sim c \sim 100$. The estimate (76) is an exception, in that it produces almost 4-digit accuracy at $m \sim c \sim 10$ (even though its convergence is only first order).
3. The obtained expansions tend to be highly satisfactory as "ballpark" estimates, or when initial approximations are needed for some iterative processes (Newton, inverse power with shifts, etc.). The authors have already used some of them for this purpose.
4. In all cases, convergence rates predicted by the estimates (64), (67), (68), (70), (72), (73), (74), (76) are confirmed numerically.
5. The expansions (64), (67), (68), (70) of Section 3.2 have been obtained via the inverse power method followed by the expansion of the result into a power series in c/m (with some additional analysis in the case of (70)). Thus, convergence rates of expansions in Section 3.2 are fairly high and can be easily improved (in fact, some of them were obtained with higher orders and truncated in order to save space and improve readability).
6. In contrast, the expansions (72), (73), (74), (76) of Section 5 have been obtained via a WKB analysis of Eq. (2) and converge slowly; as often happens in the WKB environment, higher order terms can be obtained, but tend to require fairly involved algebraic manipulation.

7. Table 9 indicates that whenever $|\lambda_m^c| \in [10^{-34}, 10^{-3}]$, the approximation (73) to λ_m^c is quite accurate, even for $c = 1$. Because of this relative universality, we found the approximation (76) to be convenient in many practical situations.

7. Conclusions

In this paper, we continue to develop asymptotic formulae for the approximation of certain prolate spheroidal wave functions; in this sense, this paper can be viewed as a sequel to [14,15]. We investigate the behavior of PSWFs in two regimes: when the ratio c/m decays and when both c and m grow, but the ratio c/m stays bounded. Both the regions of validity and the accuracies of the presented expansions are illustrated with numerical examples.

While our results are restricted to PSWFs representing band-limited functions in one dimension, they are easily extended to PSWFs in two and three dimensions (see [10]), and to the discrete version of PSWFs (see [12]).

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