1.) Diffferentiate each of the following. DO NOT SIMPLIFY answers.

a.) $y = e^{x^2}\cos^3(5x)$

$$Y' = e^{x^2}\cdot 3\cos^2(5x)\cdot -\sin(5x)\cdot 5$$
$$+ e^{x^2}\cdot 2x\cdot\cos^3(5x)$$

b.) $y = \arcsin(2^x + \log x)$

$$Y' = \frac{1}{\sqrt{1-(2^x+\log x)^2}}\cdot\left\{2^x\cdot\ln 2 + \frac{1}{x}\cdot\log_{10} e\right\}$$

c.) $y = x^{\ln x}$ $\rightarrow$ $\ln Y = \ln x\cdot\ln x = (\ln x)^2 \rightarrow$

$$\frac{1}{Y}Y' = 2(\ln x)\cdot\frac{1}{x} \rightarrow Y' = x^{\ln x}\cdot\frac{2\ln x}{x}$$

b.) $f(x) = \dfrac{3x+7}{\sqrt{x^4+1}}$

$$f'(x) = \frac{\sqrt{x^4+1}\cdot(3) - (3x+7)\cdot\frac{1}{2}(x^4+1)^{-1/2}\cdot 4x^3}{x^4+1}$$

c.) $g(x) = \sec^3(\tan^{-4}(x^{1/5}))$

$$g'(x) = 3\sec^2(\tan^{-4}(x^{1/5}))\cdot\sec(\tan^{-4}(x^{1/5}))\tan(\tan^{-4}(x^{1/5}))\cdot$$
$$-4\tan^{-5}(x^{1/5})\cdot\sec^2(x^{1/5})\cdot\frac{1}{5}x^{-4/5}$$

3.) The manager of the Economy Motel charges \$30 per room and rents ~~50~~ 48 rooms each night. For each \$5 increase in room charge four (4) fewer rooms are rented. What charge per room will maximize the total amount of money the manager will make in one night ?

Let x : # of \$5 increases,

$$\text{max. } T = \underset{\substack{\uparrow \\ \text{charge per room}}}{(30+5x)}\underset{\substack{\uparrow \\ \text{\# of rooms}}}{(\overset{48}{\cancel{50}}-4x)} \longrightarrow$$

$$T' = (30+5x)(-4) + (5)(\overset{48}{\cancel{50}}-4x)$$

$$= -120 - 20x + \underset{240}{\cancel{250}} - 20x$$

$$= 120 - 40x = 0$$

$$\begin{array}{ccc} + & 0 & - \\ \hline & x=3 & \end{array} \quad T'$$

charge : \$45

rooms : 36

\$: 1620

4.) Use the limit definition of derivative to differentiate $f(x) = \dfrac{x^2}{x+1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)^2}{x+h+1} - \frac{x^2}{x+1}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{(x^2+2hx+h^2)(x+1)-(x+h+1)x^2}{(x+h+1)\ (x+1)} \cdot \frac{1}{h}$$

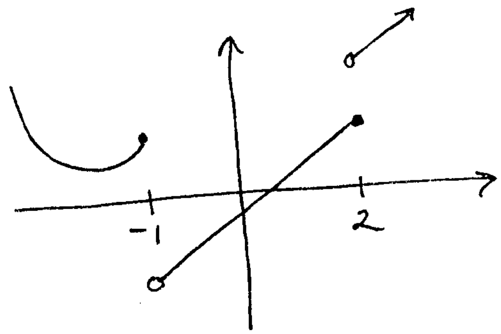
$$= \lim_{h\to 0} \frac{\cancel{x^3}+2hx^2+h^2x+\cancel{x^2}+2hx+h^2-\cancel{x^3}-hx^2-\cancel{x^2}}{(x+h+1)(x+1)\cdot h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x^2+hx+2x+h-x^2)}{(x+h+1)(x+1)\cdot\cancel{h}}$$

$$= \frac{x^2+2x}{(x+1)^2}$$

5.) Use limits to determine the values of the constants A and B so that the following function is continuous for all values of x .

$$f(x) = \begin{cases} Bx^2 + Ax, & \text{if } x \le -1 \\ 2B - Ax, & \text{if } -1 < x \le 2 \\ x + 3, & \text{if } x > 2 \; . \end{cases}$$



$$\lim_{x \to -1^-} (Bx^2 + Ax) = \lim_{x \to -1^+} (2B - Ax)$$

$$\longrightarrow B - A = 2B + A \longrightarrow \boxed{B = -2A} \; ;$$

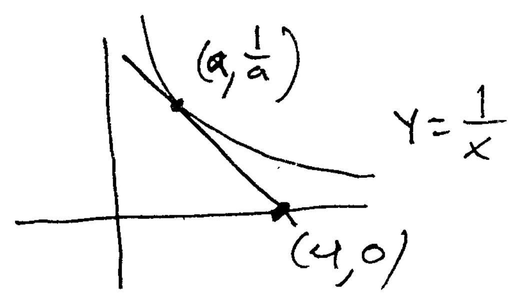
$$\lim_{x \to 2^-} (2B - Ax) = \lim_{x \to 2^+} (x + 3)$$

$$\longrightarrow \boxed{2B - 2A = 5} \longrightarrow 2(-2A) - 2A = 5 \longrightarrow$$

$$-6A = 5 \longrightarrow A = -5/6 \; , \; B = 5/3$$

6.) Find all points (x, y) on the graph of $y = \dfrac{1}{x}$ with tangent lines passing through the point (4, 0).

$$y' = \frac{-1}{x^2}$$



$$\text{slope}: \frac{\frac{1}{a} - 0}{a - 4} = \frac{-1}{a^2} \longrightarrow$$

$a = -a + 4 \longrightarrow a = 2$ so pt. is

$(2, \frac{1}{2})$.

7.) The radius of a sphere is measured with an absolute percentage error of at most 4%. Use differentials to estimate the maximum absolute percentage error in computing the volume of the sphere. ($V = \frac{4}{3}\pi r^3$.)

$V' = 4\pi r^2$ and $\frac{|\Delta r|}{r} \leq 4\%$, estimate

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta r|}{V} = \frac{|\cancel{4}\cancel{\pi}\cancel{r^2} \cdot \Delta r|}{\cancel{\frac{4}{3}}\cancel{\pi} r^{\cancel{3}}}$$

$$= 3\,\frac{|\Delta r|}{r} \leq 3(4\%) = 12\%.$$

8.) Consider the equation $27 - x^3 = \sin x$.

a.) Use the Intermediate Value Theorem to verify that the equation is solvable. Let $f(x) = 27 - x^3 - \sin x = 0$ and $m = 0$;

f is cont. since difference of continuous functions; consider interval $[2, \pi]$: $f(2) = 19 - \sin 2 > 0$ and $f(\pi) = 27 - \pi^3 - \overset{0}{\cancel{\sin \pi}} < 0$ so $m = 0$ is between $f(2)$ and $f(\pi)$. Thus, by IMVT there is at least one value c, $2 \leq c \leq \pi$, satisfying $f(c) = 0$, i.e.,

$$27 - c^3 - \sin c = 0.$$

b.) Use Newton's method to estimate the value of the solution of the equation to three decimal places. $f'(x) = -3x^2 - \cos x$ so

$$x_{n+1} = x_n - \frac{27 - x_n^3 - \sin x_n}{-3x_n^2 - \cos x_n} = \frac{\sin x_n - x_n \cos x_n - 2x_n^3 - 27}{-3x_n^2 - \cos x_n}$$

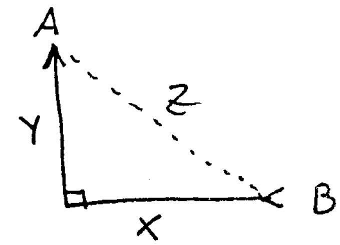
$x_1 = 3$

$x_2 = 2.99457$

$x_3 = 2.99456$

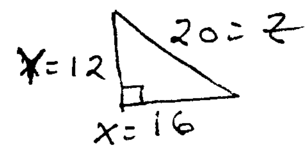
9.) Car B is 34 miles directly east of car A and begins moving west at 90 mph. At the same moment car A begins moving north at 60 mph.

a.) At what rate is the distance between the cars changing after $t = \frac{1}{5}$ hr. ?



$\frac{dY}{dt} = 60$ mph, $\frac{dX}{dt} = -90$ mph, find

$\frac{dZ}{dt}$ when $t = \frac{1}{5}$ hr $\rightarrow X = 12$ mi,
$Y = 16$ mi. :

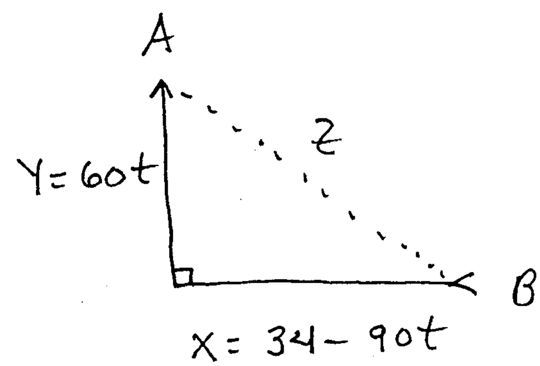


$X^2 + Y^2 = Z^2 \rightarrow 2X\frac{dX}{dt} + 2Y\frac{dY}{dt} = 2Z\frac{dZ}{dt}$

$\rightarrow (16)(-90) + (12)(60) = (20)\frac{dZ}{dt}$

$\rightarrow \frac{dZ}{dt} = -36$ mph

b.) What is the minimum distance between the cars and at what time t does the minimum distance occur ?

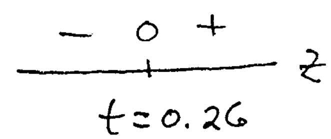


Let t be time,
minimize distance

$Z = \sqrt{(60t)^2 + (34-90t)^2} \rightarrow$

$\frac{dZ}{dt} = \frac{1}{2}(\sim)^{-\frac{1}{2}} \cdot [2(60t)60$
$+ 2(34-90t)\cdot(-90)] = 0 \rightarrow$

$360t - 306 + 810t = 0 \rightarrow t \approx 0.26$ hrs.



and min. $Z \approx 18.86$ mi.

10.) Find the slope and concavity of the graph $xy + y^2 = 3x + 1$ at the point $(0, -1)$.

$\xrightarrow{D} XY' + Y + 2YY' = 3 \rightarrow Y' = \dfrac{3-Y}{X+2Y}$

and $X=0,\ Y=-1 \rightarrow Y' = \dfrac{4}{-2} = -2 = \text{slope}$;

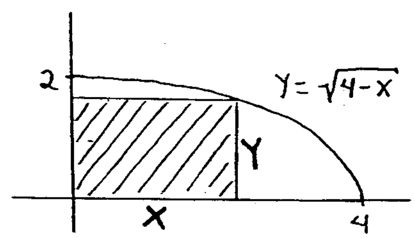
$$Y'' = \frac{(X+2Y)(-Y') - (3-Y)(1+2Y')}{(X+2Y)^2}$$

$$= \frac{(-2)(2) - (4)(-3)}{(-2)^2} = 2 = Y'' \text{ so}$$

concave up.

11.) Consider all rectangles in the first quadrant inscribed in such a way that their bases lie on the x-axis with the top corner on the graph of $y = \sqrt{4-x}$. Find the length and width of the rectangle of maximum area.

max. area



$A = XY = X\sqrt{4-X} \longrightarrow$

$A' = X \cdot \frac{1}{2}(4-X)^{-1/2}(-1) + \sqrt{4-X}$

$= \dfrac{-X}{2\sqrt{4-X}} + \dfrac{\sqrt{4-X}}{1} = \dfrac{-X + 2(4-X)}{2\sqrt{4-X}}$

$= \dfrac{8-3X}{2\sqrt{4-X}} = 0$

$+ \quad 0 \quad -$ A'

$X = 8/3$

$A = \dfrac{16}{3\sqrt{3}}$ $\quad Y = \dfrac{2}{\sqrt{3}}$

12.) Consider the function $f(x) = x\, e^{\left(\frac{-x}{2}\right)}$. Determine where f is increasing, decreasing, concave up, and concave down. Identify all relative and absolute extrema, inflection points, x- and y-intercepts, and vertical and horizontal asymptotes. Sketch the graph. You may assume that $f'(x) = \left(1 - \frac{x}{2}\right) e^{\left(\frac{-x}{2}\right)}$ and $f''(x) = \left(\frac{x}{4} - 1\right) e^{\left(\frac{-x}{2}\right)}$.

$+ \quad 0 \quad -$ f'

$- \quad 0 \quad +$ f''

abs. max. $\begin{cases} X = 2 \\ Y = \frac{2}{e} \end{cases}$

infl. pt. $\begin{cases} X = 4 \\ Y = \frac{4}{e^2} \end{cases}$

$X = 0 : Y = 0$ and $Y = 0 : X = 0$

f is $\uparrow$ for $X < 2$

f is $\downarrow$ for $X > 2$

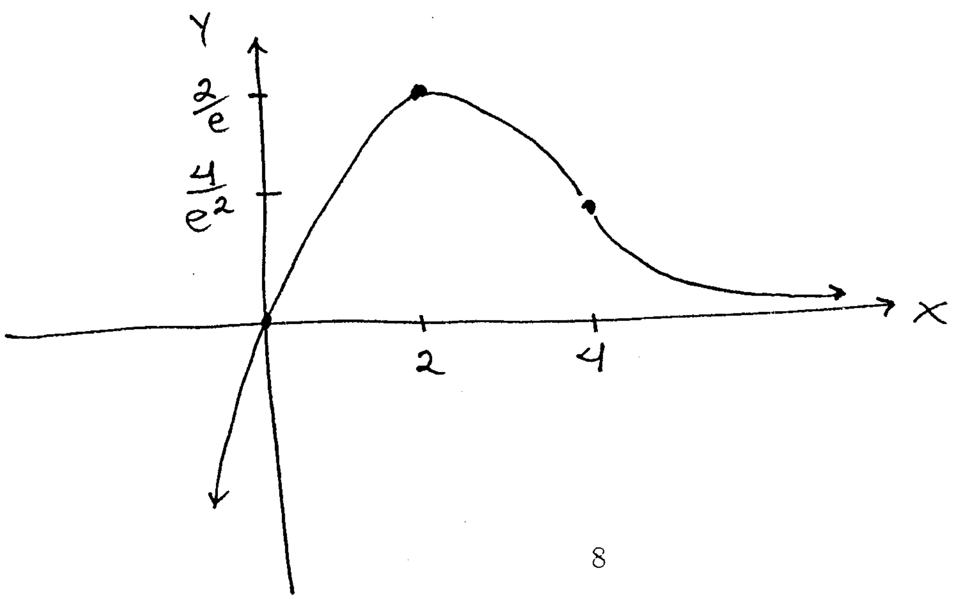
f is $\cup$ for $X > 4$

f is $\cap$ for $X < 4$

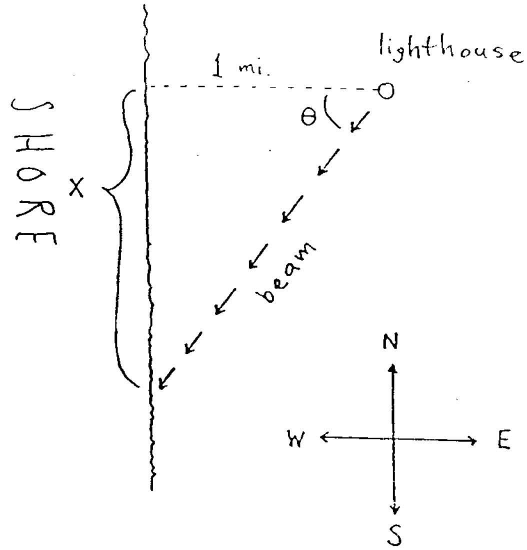
$$\lim_{X \to -\infty} x e^{-\frac{x}{2}} = -\infty \cdot \infty = -\infty$$

$$\lim_{X \to +\infty} x e^{-\frac{x}{2}} = \lim_{X \to +\infty} \frac{x}{e^{x/2}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{X \to +\infty} \frac{1}{\frac{1}{2} e^{x/2}} = \text{"}\frac{1}{\infty}\text{"} = 0$$

so horizontal asymptote $Y = 0$



13.) A lighthouse sits one (1) mile offshore with a light beam turning counter-clockwise at the rate of ten (10) revolutions per minute. How fast is the light beam racing down the shoreline when the beam strikes a point on the shore twelve miles south of the nearest point on the shore ? ~~Assume there are 5280 feet per mile and~~ write your final answer in MILES PER HOUR.

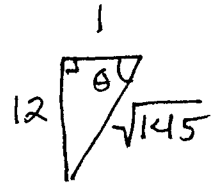


$$\frac{d\theta}{dt} = \frac{10\ \cancel{\text{rev.}}}{\text{min.}} \cdot \frac{2\pi\ \text{rad.}}{\cancel{\text{rev.}}} = \frac{20\pi\ \text{rad.}}{\text{min.}};$$

find $\frac{dx}{dt}$ when $x = 12$ mi :

$$\tan\theta = \frac{x}{1} = x \xrightarrow{D_t}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$$



$$\rightarrow \left(\sqrt{145}\right)^2 \cdot (20\pi) = \frac{dx}{dt} \rightarrow$$

$$\frac{dx}{dt} = 9110.6\ \frac{\text{mi.}}{\cancel{\text{min.}}} \cdot \frac{60\ \cancel{\text{min.}}}{1\ \text{hr.}}$$

$= 546,660$ mph

14.) Evaluate the following limits.

a.) $$\lim_{x\to 0} \frac{x\sin x}{(\arctan x)^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{x\cos x + \sin x}{2\arctan x \cdot \frac{1}{1+x^2}}$$

$$= \lim_{x\to 0} \frac{(x\cos x + \sin x)(x^2+1)}{2\arctan x}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{(x\cos x + \sin x)(2x) + (-x\sin x + \cos x + \cos x)(x^2+1)}{2 \cdot \frac{1}{1+x^2}}$$

$$= \frac{2}{2} = 1$$

b.) $\lim_{x\to 0+} x \ln x$ $=$ "$0 \cdot -\infty$" $= \lim_{x\to 0+} \dfrac{\ln x}{\frac{1}{x}}$

$\overset{\text{"}\frac{-\infty}{\infty}\text{"}}{=} \lim_{x\to 0+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0+} (-x) = 0$

c.) $\lim_{x\to\infty} (x^3+4)^{\frac{1}{x}}$ $=$ "∞^0"

$\ln\left(\lim_{x\to\infty} (x^3+4)^{\frac{1}{x}}\right) = \lim_{x\to\infty} \ln(x^3+4)^{1/x}$

$= \lim_{x\to\infty} \dfrac{\ln(x^3+4)}{x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \dfrac{\frac{3x^2}{x^3+4}}{1}$

$= \lim_{x\to\infty} \dfrac{3x^2 \cdot \frac{1}{x^2}}{(x^3+4)\cdot\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{3}{x+\frac{4}{x^2}}$

$=$ "$\dfrac{3}{\infty+0}$" $= 0$ so $\lim_{x\to\infty} (x^3+4)^{1/x} = 1$

Each of the following three EXTRA CREDIT PROBLEMS is worth These problems are OPTIONAL.

1.) Show that $\log_B C = \dfrac{\ln C}{\ln B}$.

Let $\log_B C = x \rightarrow$

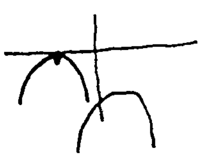
$B^x = C \rightarrow \ln B^x = \ln C \rightarrow x \ln B = \ln C \rightarrow$

$x = \dfrac{\ln C}{\ln B} \rightarrow \log_B C = \dfrac{\ln C}{\ln B}$

2.) Find all values of K for which the the function $f(x) = -x^3 + Kx + x^2$ is NOT one-to-one.

f is 1-1 if $f'(x) \leq 0 \rightarrow f'(x) = -3x^2 + K + 2x \leq 0 \rightarrow$

$-3x^2 + 2x + K \leq 0 \qquad x = \dfrac{-2 \pm \sqrt{4 + 12K}}{-6}$



so $4 + 12K \leq 0 \rightarrow K \leq \frac{-1}{3}$

NOT 1-1 : $K > \frac{-1}{3}$