Worksheet 3

1.) a.) Prove that $\lim_{x \to 10} (3x+5) = 35$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x - 10| < \delta$ then $|f(x) - 35| < \varepsilon$. Begin with $|f(x) - 35| < \varepsilon$ and "solve" for $|x-10|$. Then

$$
\begin{aligned}
|f(x) - 35| < \varepsilon \text{ iff } & |(3x+5) - 35| < \varepsilon \\
\text{iff } & |3x - 30| < \varepsilon \\
\text{iff } & 3|x-10| < \varepsilon \\
\text{iff } & |x-10| < \varepsilon/3 .
\end{aligned}
$$

Choose $\delta = \varepsilon/3$. Thus, it follows that if $0 < |x-10| < \varepsilon/3$, then $|f(x) - 35| < \varepsilon$. This completes the proof.

b.) Prove that $\lim_{x \to -\frac{3}{2}} (1-4x) = 7$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x - (-\frac{3}{2})| < \delta$ then $|f(x) - 7| < \varepsilon$, i.e., if $0 < |x + \frac{3}{2}| < \delta$ then $|f(x) - 7| < \varepsilon$. Begin with $|f(x) - 7| < \varepsilon$ and "solve" for $|x + \frac{3}{2}|$. Then

$$|f(x) - 7| < \varepsilon \text{ iff } |(1-4x) - 7| < \varepsilon$$

$$\text{iff} \quad |-6-4x| < \varepsilon$$
$$\text{iff} \quad \left|(-4)\left(\tfrac{3}{2}+x\right)\right| < \varepsilon$$
$$\text{iff} \quad 4\,|x+3/2| < \varepsilon$$
$$\text{iff} \quad |x+3/2| < \varepsilon/4 .$$

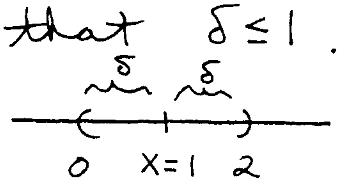
Choose $\delta = \varepsilon/4$. Thus, if $0 < |x+3/2| < \delta$, it follows that $|f(x)-7| < \varepsilon$. This completes the proof.

c.) Prove that $\lim\limits_{x \to 1} (x^2+3) = 4$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x-1| < \delta$ then $|f(x)-4| < \varepsilon$. Begin with $|f(x)-4| < \varepsilon$ and solve for $|x-1|$. Then

$$|f(x)-4| < \varepsilon \quad \text{iff} \quad |(x^2+3)-4| < \varepsilon$$
$$\text{iff} \quad |x^2-1| < \varepsilon$$
$$\text{iff} \quad |(x-1)(x+1)| < \varepsilon$$
$$\text{iff} \quad |x-1|\,|x+1| < \varepsilon .$$

We must "eliminate" the term $|x+1|$. Assume that $\delta \le 1$. Then $0 < x < 2$ and



$$1 < |x+1| < 3 .$$ Thus,

$$|x-1|\,|x+1| < |x-1| \cdot (3) < \varepsilon$$
$$\text{iff} \quad |x-1| < \varepsilon/3 .$$

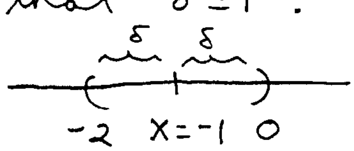
Choose $\delta = \min\{1, \varepsilon/3\}$. Thus, if $0 < |x-1| < \delta$ then $|f(x) - 4| < \varepsilon$. This completes the proof.

d.) Prove that $\lim_{x \to -1} (x^2+3) = 4$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x-(-1)| < \delta$ then $|f(x) - 4| < \varepsilon$, i.e., if $0 < |x+1| < \delta$ then $|f(x)-4| < \varepsilon$. Begin with $|f(x) - 4| < \varepsilon$ and "solve" for $|x+1|$. Then

$$\begin{aligned} |f(x)-4| < \varepsilon &\text{ iff } |(x^2+3) - 4| < \varepsilon \\ &\text{ iff } |x^2 - 1| < \varepsilon \\ &\text{ iff } |(x-1)(x+1)| < \varepsilon \\ &\text{ iff } |x-1|\,|x+1| < \varepsilon . \end{aligned}$$

We must "eliminate" the term $|x-1|$. Assume that $\delta \le 1$. Then $-2 < x < 0$ and



$1 < |x-1| < 3$. Thus

$$|x-1|\,|x+1| < (3)\,|x+1| < \varepsilon$$

$$\text{iff} \quad |x+1| < \varepsilon/3 .$$

Choose $\delta = \min\{1, \varepsilon/3\}$. Thus, if $0 < |x+1| < \delta$ then $|f(x)-4| < \varepsilon$. This completes the proof.

e.) Prove that $\lim_{x \to 3} \frac{2}{x+3} = \frac{1}{3}$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x-3| < \delta$ then $|f(x) - \frac{1}{3}| < \varepsilon$. Begin with $|f(x) - \frac{1}{3}| < \varepsilon$ and solve for $|x-3|$. Then

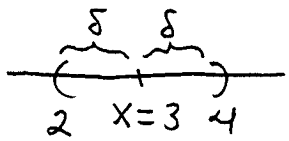
$$|f(x) - \tfrac{1}{3}| < \varepsilon \text{ iff } \left| \frac{2}{x+3} - \frac{1}{3} \right| < \varepsilon$$

$$\text{iff } \left| \frac{6-(x+3)}{3(x+3)} \right| < \varepsilon$$

$$\text{iff } \left| \frac{3-x}{3(x+3)} \right| < \varepsilon$$

$$\text{iff } \frac{1}{3} \frac{|x-3|}{|x+3|} < \varepsilon .$$

We must "eliminate" the term $|x+3|$. Assume that $\delta \le 1$. Then $2 < x < 4$



and $5 < |x+3| < 7$ so that $\frac{1}{7} < \frac{1}{|x+3|} < \frac{1}{5}$. Thus,

$$\frac{1}{3} |x-3| \cdot \frac{1}{|x+3|} < \frac{1}{3} |x-3| \cdot \left(\frac{1}{5}\right) < \varepsilon$$

$$\text{iff } \frac{1}{15} |x-3| < \varepsilon$$

$$\text{iff } |x-3| < 15\varepsilon .$$

Choose $\delta = \min\{1, 15\varepsilon\}$. Thus, if $0 < |x-3| < \delta$ then $|f(x) - \frac{1}{3}| < \varepsilon$. This completes the proof.

f.) Prove that $\lim\limits_{x \to -6} \dfrac{x+4}{2-x} = \dfrac{-1}{4}$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x-(-6)| < \delta$ then $|f(x) - (\frac{-1}{4})| < \varepsilon$, i.e., if $0 < |x+6| < \delta$ then $|f(x) + \frac{1}{4}| < \varepsilon$. Begin with $|f(x) + \frac{1}{4}| < \varepsilon$ and "solve" for $|x+6|$. Then

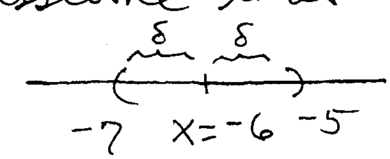
$$|f(x) + \tfrac{1}{4}| < \varepsilon \text{ iff } \left| \frac{x+4}{2-x} + \frac{1}{4} \right| < \varepsilon$$

$$\text{iff } \left| \frac{4(x+4) + (2-x)}{4(2-x)} \right| < \varepsilon$$

$$\text{iff } \left| \frac{3x+18}{4(2-x)} \right| < \varepsilon$$

$$\text{iff } \frac{3}{4} \frac{|x+6|}{|x-2|} < \varepsilon .$$

We must "eliminate" the term $|x-2|$. Assume that $\delta \le 1$. Then $-7 < x < -5$



and $7 < |x-2| < 9$ so that $\frac{1}{9} < \frac{1}{|x-2|} < \frac{1}{7}$. Thus,

$$\frac{3}{4} \cdot |x+6| \cdot \frac{1}{|x-2|} < \frac{3}{4} |x+6| \cdot \left(\frac{1}{7}\right) < \varepsilon$$

$$\text{iff } \frac{3}{28} |x+6| < \varepsilon$$

$$\text{iff } |x+6| < \frac{28}{3} \varepsilon .$$ Choose $\delta = \min\{1, \frac{28}{3}\varepsilon\}$.

Thus, if $0 < |x+6| < \delta$ then $|f(x) + \frac{1}{4}| < \varepsilon$. This completes the proof.

9.) Prove that $\lim_{x \to 9} (\sqrt{x} + 2) = 5$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x-9| < \delta$ then $|f(x) - 5| < \varepsilon$. Begin with $|f(x)-5| < \varepsilon$ and solve for $|x-9|$. Then

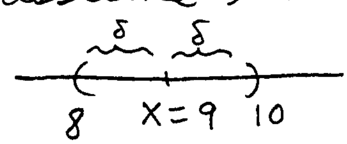
$$|f(x)-5| < \varepsilon \text{ iff } |(\sqrt{x}+2)-5| < \varepsilon$$

$$\text{iff } |\sqrt{x}-3| < \varepsilon$$

$$\text{iff } \left| (\sqrt{x}-3) \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)} \right| < \varepsilon$$

$$\text{iff } \frac{|x-9|}{|\sqrt{x}+3|} < \varepsilon .$$

We must eliminate the term $|\sqrt{x}+3|$. Assume that $\delta \leq 1$. Then $8 < x < 10$ and



$$\sqrt{8}+3 < |\sqrt{x}+3| < \sqrt{10}+3$$

so that $\frac{1}{\sqrt{10}+3} < \frac{1}{|\sqrt{x}+3|} < \frac{1}{\sqrt{8}+3}$.

Thus,

$$|x-9| \cdot \frac{1}{|\sqrt{x}+3|} < |x-9| \cdot \frac{1}{\sqrt{8}+3} < \varepsilon$$

$$\text{iff } |x-9| < (\sqrt{8}+3)\,\varepsilon .$$

Choose $\delta = \min\{1, (\sqrt{8}+3)\varepsilon\}$. Thus, if $0 < |x-9| < \delta$ then $|f(x)-5| < \varepsilon$. This completes the proof.