Section 2.6

16.) $y = \frac{x+3}{x^2-3x-10}$; $y = x+3$ and $y = x^2-3x-10$ are continuous for all values of x since they are polynomials; therefore, since $y = \frac{x+3}{x^2-3x-10}$ is the <u>quotient</u> of these functions, it is <u>continuous for all values of x</u> except where $x^2-3x-10 = (x-5)(x+2) = 0$, i.e., <u>except for $x = 5$ and $x = -2$</u>.

20.) $y = \frac{x+2}{\cos x}$; $y = x+2$ is continuous for all values of x since it is a polynomial; $y = \cos x$ is continuous for all values of x since it is a well-known trig function; therefore, since $y = \frac{x+2}{\cos x}$ is the <u>quotient</u> of these functions, it is <u>continuous for all values of x</u> except where $\cos x = 0$, i.e., <u>except for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$</u>

26.) $y = (3x-1)^{1/4}$; let $f(x) = x^{1/4}$, which is continuous for $x \geq 0$, and let $g(x) = 3x-1$, which is continuous for all values of x since it is a polynomial; since $y = (3x-1)^{1/4} = f(3x-1) = f(g(x))$ is <u>functional</u> <u>composition</u>, it is continuous for all x-values for which $3x-1 \geq 0$, i.e., for $x \geq 1/3$.

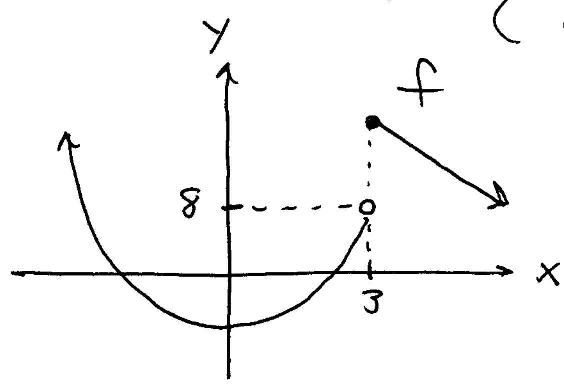
38.) $g(x) = \dfrac{x^2-16}{x^2-3x-4}$ then

$$\lim_{x \to 4} g(x) = \lim_{x \to 4} \frac{x^2-16}{x^2-3x-4}$$

$$\text{"}\tfrac{0}{0}\text{"} \lim_{x \to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}(x+1)} = \frac{8}{5}, \text{ so}$$

define $g(4) = 8/5$ and g will be continuous at $x=4$.

39.) Let $f(x) = \begin{cases} x^2-1, & \text{if } x < 3 \\ 2ax, & \text{if } x \geq 3 \end{cases}$



$y = x^2-1$ is continuous for $x < 3$ (polynomial); $y = 2ax$ is continuous for $x > 3$ (line); make f continuous at $x=3$ by forcing limits to be equal:

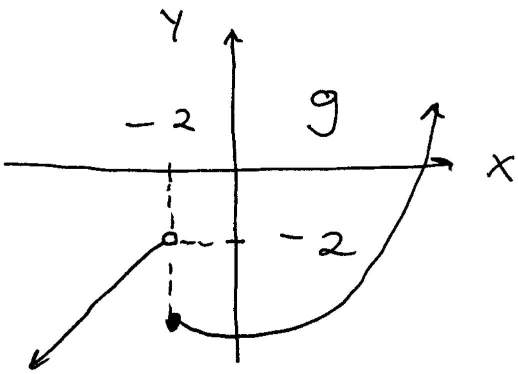
$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2-1) = 9-1 = 8,$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2ax) = 6a, \text{ so}$$

$6a = 8 \rightarrow a = 4/3$.

40.) Let $g(x) = \begin{cases} x, & \text{if } x < -2 \\ bx^2, & \text{if } x \geq -2 \end{cases}$

$Y = x$ is continuous for $x < -2$ (line), $Y = bx^2$ is continuous for $x > -2$ (parabola); make g continuous at $x = -2$ by forcing limits to be equal:



$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x = -2,$$

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} bx^2 = 4b,$$

so $4b = -2 \rightarrow b = -1/2$.

I.) Prove $x^3 = x + 2$ is solvable:

$x^3 = x + 2 \rightarrow x^3 - x - 2 = 0$, so let $\underline{f(x) = x^3 - x - 2}$ and $\underline{m = 0}$; note that $f(1) = -2 < 0$ and $f(2) = 4 > 0$ so $m = 0$ is between $f(1)$ and $f(2)$; use the interval $[1, 2]$; f is a continuous function on $[1, 2]$ since it is a polynomial. By the IMVT it follows that there is a number c, $1 \leq c \leq 2$, so that $f(c) = m$, i.e.,

$$c^3 - c - 2 = 0,$$

and the original equation is solvable.

II.) Prove $2 + \sin x = x$ is solvable:

$2 + \sin x = x \rightarrow 2 - x + \sin x = 0$ so let $\underline{f(x) = 2 - x + \sin x}$ and $\underline{m = 0}$;

f is <u>continuous</u> for all values of x since it is the sum of continuous functions ($y = 2 - x$, a line, and $y = \sin x$, a well-known trig function); note that $f(0) = 2 > 0$ and $f(\pi) = 2 - \pi - \sin\pi = 2 - \pi < 0$, so $m = 0$ is <u>between</u> $f(0)$ and $f(\pi)$; use the interval $\underline{[0, \pi]}$. By the IMVT it follows that there is a number c, $0 \le c \le \pi$, so that $f(c) = m$, i.e.,

$$2 - c + \sin c = 0,$$

and the original equation is solvable.

Math 21A
Kouba
Worksheet 2

1.) Use limits and algebra to determine the value of constants A and B so that each of the following functions is continuous for all values of x.

a.) $f(x) = \begin{cases} \dfrac{x^2-7x+6}{x-6}, & \text{if } x \neq 6 \\ A, & \text{if } x = 6. \end{cases}$

b.) $f(x) = \begin{cases} A^2x - A, & \text{if } x \geq 1 \\ 2, & \text{if } x < 1. \end{cases}$

c.) $f(x) = \begin{cases} \dfrac{A+x}{A+1}, & \text{if } x < 0 \\ Ax^3 + 3, & \text{if } x \geq 0. \end{cases}$

d.) $f(x) = \begin{cases} 3, & \text{if } x \leq 1 \\ Ax^2 + B, & \text{if } 1 < x \leq 2 \\ 5, & \text{if } x > 2. \end{cases}$

e.) $f(x) = \begin{cases} Ax - B, & \text{if } x \leq -1 \\ 2x + 3A + B, & \text{if } -1 < x \leq 1 \\ 4, & \text{if } x > 1. \end{cases}$

2.) For what x-values are the following functions continuous ? Briefly explain why using shortcuts and rules from class. Sketch the graph of each using a graphing calculator.

a.) $g(x) = \dfrac{x+1}{x^2-4}$

b.) $h(x) = \dfrac{100}{4 - \sqrt{x^2-9}}$

c.) $h(x) = \sin^3(\ln(3x-5))$

d.) $g(x) = \begin{cases} \dfrac{x^2-3x-4}{x-4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4. \end{cases}$

e.) $f(x) = \begin{cases} \dfrac{x^3+1}{x^2-1}, & \text{if } x \neq 1, -1 \\ -3/2, & \text{if } x = -1 \\ 3, & \text{if } x = 1\ . \end{cases}$

Worksheet 2 Solutions

1.) a.) Since $\lim_{x\to 6} f(x) = \lim_{x\to 6} \frac{x^2-7x+6}{x-6}$

$\overset{\text{"}0\text{"}}{=} \lim_{x\to 6} \frac{\cancel{(x-6)}(x-1)}{\cancel{(x-6)}} = 5$, choosing $\boxed{a=5}$

makes f continuous at $x=6$ (It's already continuous for $x \neq 6$.)

b.) f is continuous for $x<1$ and for $x>1$. We must make f continuous at $x=1$:

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (a^2x-a) = a^2-a$ and

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2) = 2$, thus $a^2-a=2 \rightarrow$

$a^2-a-2=0 \rightarrow (a-2)(a+1)=0 \rightarrow \boxed{a=2}$ or $\boxed{a=-1}$

c.) f is continuous for $x<0$ (so long as $a \neq -1$) and for $x>0$. We must make f continuous at $x=0$:

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (ax^3+3) = 3$ and

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{a+x}{a+1} = \frac{a}{a+1}$, thus $\frac{a}{a+1} = 3 \rightarrow$

$a = 3a+3 \rightarrow -3 = 2a \rightarrow a = \frac{-3}{2}$.

d.) f is continuous for $x<1$, for $1<x<2$, and for $x>2$. We must make f continuous at $x=1$ and at $x=2$:

at $x=1$: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (ax^2+b) = a+b$ and

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3) = 3$, thus $\boxed{a+b=3}$;

at $x=2$: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (5) = 5$ and

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (ax^2+b) = 4a+b$, so $\boxed{4a+b=5}$;

thus $\left.\begin{aligned} a+b&=3 \\ 4a+b&=5 \end{aligned}\right\}$ $b=3-a$ $\rightarrow 4a+(3-a)=5 \rightarrow$

$3a=2 \rightarrow$ $a=2/3$ and $b=\frac{7}{3}$.

e.) f is continuous for $x<-1$, for $-1<x<1$, and for $x>1$. We must make f continuous at $x=-1$ and $x=1$:

at $x=-1$: $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (2x+3a+b) = 3a+b-2$ and

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (ax-b) = -a-b$, so $\boxed{3a+b-2=-a-b}$;

at $x=1$: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4) = 4$ and

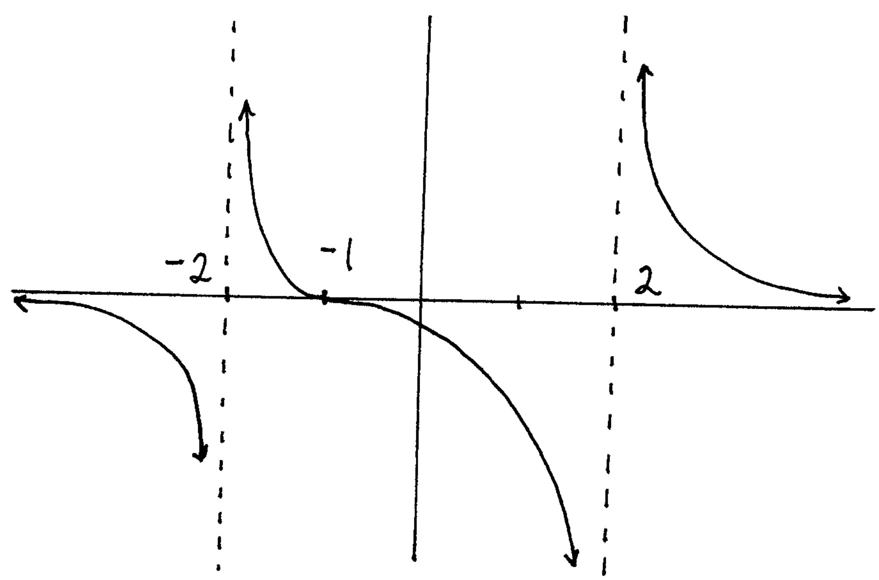
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x+3a+b) = 2+3a+b$, so $\boxed{2+3a+b=4}$;

thus, $\left.\begin{array}{l} 3a+b-2=-a-b \\ 2+3a+b=4 \end{array}\right\}$ $\left.\begin{array}{l} 4a+2b=2 \\ 3a+b=2 \end{array}\right\}$ $b=2-3a$

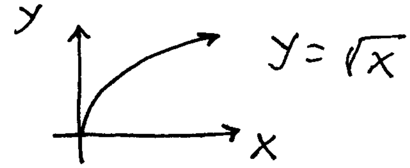
$\rightarrow 4a+2(2-3a)=2 \rightarrow 4a+4-6a=2 \rightarrow$

$2=2a \rightarrow a=1$ and $b=-1$

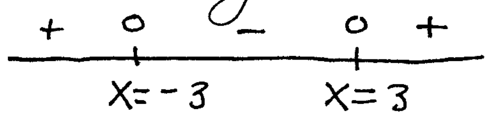
2.) a.) $y=x+1$ and $y=x^2-4$ are continuous for all values of x (since they are polynomials), so $g(x)=\dfrac{x+1}{x^2-4}$ is continuous for all values of x (quotient of continuous functions) except where $x^2-4=(x-2)(x+2)=0$, i.e., except for $x=2$ and $x=-2$.



b.) $y=x^2-9$ and $y=100$ are continuous for all values of x (since they are polynomials); $y=\sqrt{x}$ is a well



known continuous function for $x \geq 0$; let $f(x) = \sqrt{x}$ and $g(x) = x^2 - 9$, then $\sqrt{x^2-9} = f(g(x))$ is continuous (composition of continuous functions) so long as $x^2 - 9 \geq 0$, i.e., $(x-3)(x+3) \geq 0$,



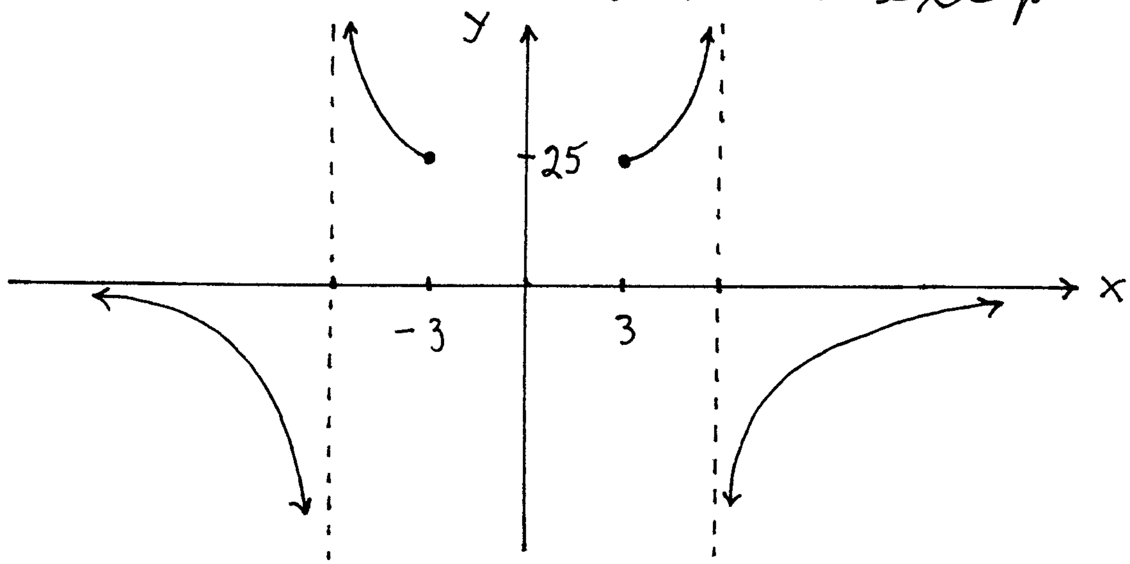
i.e., for $x \geq 3$ and $x \leq -3$;

$y = 4$ is continuous for all values of x, so that $y = 4 - \sqrt{x^2-9}$ is continuous (difference of continuous functions) for $x \geq 3$ and $x \leq -3$; finally,

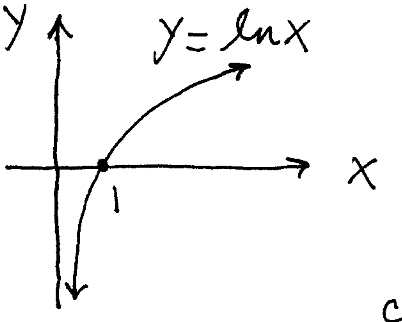
$h(x) = \dfrac{100}{4 - \sqrt{x^2-9}}$ is continuous (quotient of continuous functions) for $x \geq 3$ and $x \leq -3$ so long as $4 - \sqrt{x^2-9} \neq 0$;

$4 - \sqrt{x^2-9} = 0 \Rightarrow 4 = \sqrt{x^2-9} \Rightarrow 16 = x^2 - 9 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$; thus,

$h(x) = \dfrac{100}{4 - \sqrt{x^2-9}}$ is continuous for $x \geq 3$ and $x \leq -3$ except $x = \pm 5$.



c.) $y = 3x-5$ and $y = x^3$ are continuous for all values of x (since they are polynomials), and $y = \sin x$ is a well known function continuous for all values of x; $y = \ln x$ is a well known function continuous for $x > 0$;



let $f(x) = \ln x$ and $g(x) = 3x-5$, then $\ln(3x-5) = f(g(x))$ is continuous (composition of continuous functions) so long as $3x-5 > 0$, i.e., for $x > 5/3$; let $k(x) = x^3$ and $l(x) = \sin x$ then $h(x) = \sin^3(\ln(3x-5)) = k(l(f(g(x))))$ is continuous (composition of continuous functions) for $x > 5/3$.

For graph of function try the following ranges for x:

1. $5/3 < x \leq 1000$
2. $5/3 < x \leq 100$
3. $5/3 < x \leq 10$
4. $5/3 < x \leq 2$
5. $5/3 < x \leq 1.75$
6. $5/3 < x \leq 1.68$
7. $5/3 < x \leq 1.668$
8. $5/3 < x \leq 1.6668$

d.) $g(x) = \begin{cases} \dfrac{x^2-3x-4}{x-4} & , \text{if } x \neq 4 \\ 5 & , \text{if } x = 4 \end{cases}$

$= \begin{cases} \dfrac{(x-4)(x+1)}{x-4} & , \text{if } x \neq 4 \\ 5 & , \text{if } x = 4 \end{cases}$

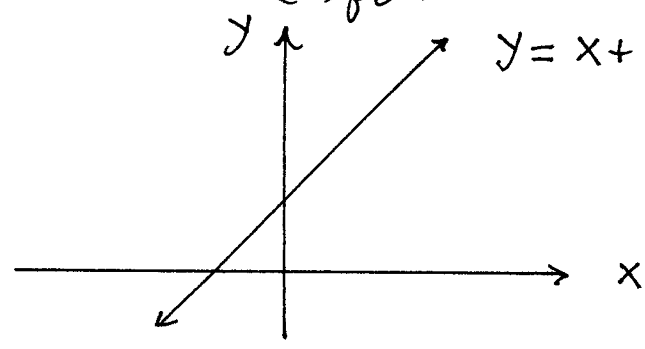
$= \begin{cases} x+1 & , \text{if } x \neq 4 \\ 5 & , \text{if } x = 4 \end{cases}$;

i.) $g(4) = 5$

ii.) $\lim_{x \to 4} g(x) = \lim_{x \to 4} (x+1) = 4+1 = 5$

iii) $\lim_{x \to 4} g(x) = g(4)$;

thus g is continuous at $x = 4$; since $y = x+1$ is continuous for $x \neq 4$ (since it is a polynomial), g is continuous for all values of x.



e.) $f(x) = \begin{cases} \dfrac{x^3+1}{x^2-1} & , \text{if } x \neq 1, -1 \\ -3/2 & , \text{if } x = -1 \\ 3 & , \text{if } x = 1 \end{cases}$

$y = x^3+1$ and $y = x^2-1$ are continuous for all values of x (since they are polynomials), so $y = \frac{x^3+1}{x^2-1}$ is continuous for all values of x except where $x^2-1=0$, i.e., except for $x = \pm 1$;

<u>check $x=1$</u>: i.) $f(1) = 3$, ii.) $\lim_{x\to 1} f(x)$

$= \lim_{x\to 1} \frac{x^3+1}{x^2-1} = \text{"}\frac{2}{0^{\pm}}\text{"} = \pm\infty$ so $\lim_{x\to 1} f(x)$

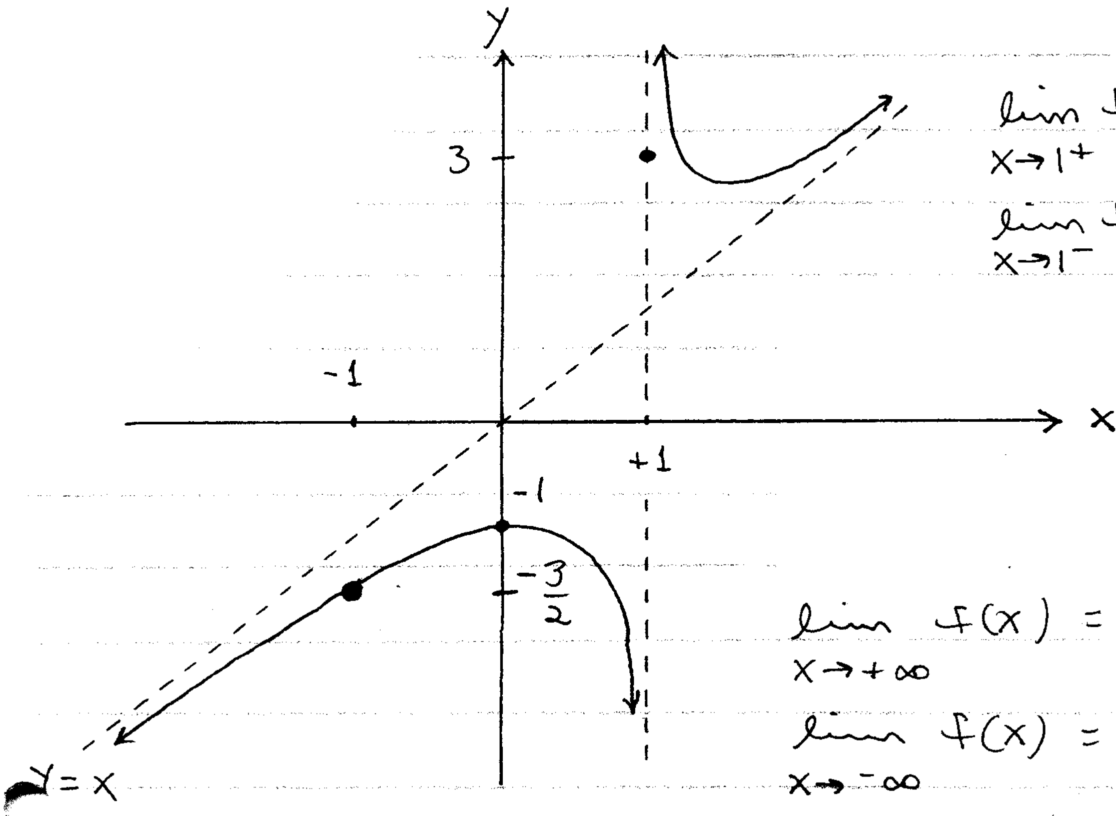
does NOT exist and f is <u>NOT cont. at $x=1$</u>;

<u>check $x=-1$</u>: i.) $f(-1) = \frac{-3}{2}$, ii.) $\lim_{x\to -1} f(x)$

$= \lim_{x\to -1} \frac{x^3+1}{x^2-1} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to -1} \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}(x-1)} = \frac{3}{-2} = \frac{-3}{2}$

and iii.) $f(-1) = \lim_{x\to -1} f(x)$ <u>so that</u>

f is continuous at $x=-1$; thus, f is continuous for all x-values except $x=1$.



$\lim_{x\to 1^+} f(x) = \text{"}\frac{2}{0^+}\text{"} = +\infty,$

$\lim_{x\to 1^-} f(x) = \text{"}\frac{2}{0^-}\text{"} = -\infty,$

$\lim_{x\to +\infty} f(x) = +\infty,$

$\lim_{x\to -\infty} f(x) = -\infty$