

Math 21A
Kouba
Worksheet **3**

1.) Use the precise epsilon/delta definition of limit to prove the following statements. These are writing exercises like those done in class. You must be clear, concise, and organized.

a.) $\lim_{x \rightarrow 10} (3x + 5) = 35$

b.) $\lim_{x \rightarrow -3/2} (1 - 4x) = 7$

c.) $\lim_{x \rightarrow 1} (x^2 + 3) = 4$

d.) $\lim_{x \rightarrow -1} (x^2 + 3) = 4$

e.) $\lim_{x \rightarrow 3} \frac{2}{x + 3} = \frac{1}{3}$

f.) $\lim_{x \rightarrow -6} \frac{x + 4}{2 - x} = \frac{-1}{4}$

g.) $\lim_{x \rightarrow 9} (\sqrt{x} + 2) = 5$

Worksheet 3

1.) a.) Prove that $\lim_{x \rightarrow 10} (3x+5) = 35$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x - 10| < \delta$ then $|f(x) - 35| < \varepsilon$. Begin with $|f(x) - 35| < \varepsilon$ and "solve" for $|x - 10|$. Then

$$\begin{aligned} |f(x) - 35| < \varepsilon &\text{ iff } |(3x+5) - 35| < \varepsilon \\ &\text{ iff } |3x - 30| < \varepsilon \\ &\text{ iff } 3|x - 10| < \varepsilon \\ &\text{ iff } |x - 10| < \varepsilon/3. \end{aligned}$$

Choose $\delta = \varepsilon/3$. Thus, it follows that if $0 < |x - 10| < \varepsilon/3$, then $|f(x) - 35| < \varepsilon$. This completes the proof.

b.) Prove that $\lim_{x \rightarrow -\frac{3}{2}} (1-4x) = 7$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x - (-\frac{3}{2})| < \delta$ then $|f(x) - 7| < \varepsilon$, i.e., if $0 < |x + \frac{3}{2}| < \delta$ then $|f(x) - 7| < \varepsilon$. Begin with $|f(x) - 7| < \varepsilon$ and "solve" for $|x + \frac{3}{2}|$. Then

$$|f(x) - 7| < \varepsilon \text{ iff } |(1-4x) - 7| < \varepsilon$$

$$\text{iff } |-6 - 4x| < \varepsilon$$

$$\text{iff } |(-4)(\frac{3}{2} + x)| < \varepsilon$$

$$\text{iff } 4|x + \frac{3}{2}| < \varepsilon$$

$$\text{iff } |x + \frac{3}{2}| < \frac{\varepsilon}{4}.$$

Choose $\delta = \frac{\varepsilon}{4}$. Thus, if $0 < |x + \frac{3}{2}| < \delta$, it follows that $|f(x) - 7| < \varepsilon$. This completes the proof.

c.) Prove that $\lim_{x \rightarrow 1} (x^2 + 3) = 4$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x - 1| < \delta$ then $|f(x) - 4| < \varepsilon$. Begin with $|f(x) - 4| < \varepsilon$ and solve for $|x - 1|$. Then

$$|f(x) - 4| < \varepsilon \text{ iff } |(x^2 + 3) - 4| < \varepsilon$$

$$\text{iff } |x^2 - 1| < \varepsilon$$

$$\text{iff } |(x-1)(x+1)| < \varepsilon$$

$$\text{iff } |x-1||x+1| < \varepsilon.$$

We must "eliminate" the term $|x+1|$. Assume that $\delta \leq 1$. Then $0 < x < 2$ and

$$\begin{array}{c} \delta \quad \delta \\ \text{---} \\ 0 \quad x=1 \quad 2 \end{array}$$

$$1 < |x+1| < 3. \text{ Thus,}$$

$$|x-1||x+1| < |x-1| \cdot (3) < \varepsilon$$

$$\text{iff } |x-1| < \frac{\varepsilon}{3}.$$

Choose $\delta = \min \{1, \frac{\epsilon}{3}\}$. Thus, if $0 < |x-1| < \delta$ then $|f(x)-4| < \epsilon$. This completes the proof.

d.) Prove that $\lim_{x \rightarrow -1} (x^2+3) = 4$;

Let $\epsilon > 0$ be given. Determine $\delta > 0$ (which depends on ϵ) so that if $0 < |x-(-1)| < \delta$ then $|f(x)-4| < \epsilon$, i.e., if $0 < |x+1| < \delta$ then $|f(x)-4| < \epsilon$. Begin with $|f(x)-4| < \epsilon$ and "solve" for $|x+1|$. Then

$$\begin{aligned} |f(x)-4| < \epsilon &\text{ iff } |(x^2+3)-4| < \epsilon \\ &\text{ iff } |x^2-1| < \epsilon \\ &\text{ iff } |(x-1)(x+1)| < \epsilon \\ &\text{ iff } |x-1||x+1| < \epsilon. \end{aligned}$$

We must "eliminate" the term $|x-1|$. Assume that $\delta \leq 1$. Then $-2 < x < 0$ and

$$\frac{\overbrace{\delta}^{\delta} \quad \overbrace{\delta}^{\delta}}{(-2 \quad | \quad -1 \quad | \quad 0)} \quad 1 < |x-1| < 3. \quad \text{Thus}$$

$$|x-1||x+1| < (3)|x+1| < \epsilon$$

$$\text{iff } |x+1| < \frac{\epsilon}{3}.$$

Choose $\delta = \min \{1, \frac{\epsilon}{3}\}$. Thus, if $0 < |x+1| < \delta$ then $|f(x)-4| < \epsilon$. This completes the proof.

e.) Prove that $\lim_{x \rightarrow 3} \frac{2}{x+3} = \frac{1}{3}$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x-3| < \delta$ then $|f(x) - \frac{1}{3}| < \varepsilon$. Begin with $|f(x) - \frac{1}{3}| < \varepsilon$ and solve for $|x-3|$.

Then

$$|f(x) - \frac{1}{3}| < \varepsilon \quad \text{iff} \quad \left| \frac{2}{x+3} - \frac{1}{3} \right| < \varepsilon$$

$$\text{iff} \quad \left| \frac{6 - (x+3)}{3(x+3)} \right| < \varepsilon$$

$$\text{iff} \quad \left| \frac{3-x}{3(x+3)} \right| < \varepsilon$$

$$\text{iff} \quad \frac{1}{3} \frac{|x-3|}{|x+3|} < \varepsilon .$$

We must "eliminate" the term $|x+3|$.

assume that $\delta \leq 1$. Then $2 < x < 4$

$$\begin{array}{c} \delta \quad \delta \\ \underbrace{\hspace{2cm}} \\ 2 \quad x=3 \quad 4 \end{array}$$

and $5 < |x+3| < 7$ so that

$$\frac{1}{7} < \frac{1}{|x+3|} < \frac{1}{5} . \quad \text{Thus,}$$

$$\frac{1}{3} |x-3| \cdot \frac{1}{|x+3|} < \frac{1}{3} |x-3| \cdot \left(\frac{1}{5}\right) < \varepsilon$$

$$\text{iff} \quad \frac{1}{15} |x-3| < \varepsilon$$

$$\text{iff} \quad |x-3| < 15\varepsilon .$$

Choose $\delta = \min \{1, 15\varepsilon\}$. Thus, if $0 < |x-3| < \delta$ then $|f(x) - \frac{1}{3}| < \varepsilon$. This completes the proof.

f.) Prove that $\lim_{x \rightarrow -6} \frac{x+4}{2-x} = \frac{-1}{4}$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x - (-6)| < \delta$ then $|f(x) - (-\frac{1}{4})| < \varepsilon$, i.e., if $0 < |x+6| < \delta$ then $|f(x) + \frac{1}{4}| < \varepsilon$. Begin with $|f(x) + \frac{1}{4}| < \varepsilon$ and "solve" for $|x+6|$. Then

$$\begin{aligned} |f(x) + \frac{1}{4}| < \varepsilon & \text{ iff } \left| \frac{x+4}{2-x} + \frac{1}{4} \right| < \varepsilon \\ & \text{ iff } \left| \frac{4(x+4) + (2-x)}{4(2-x)} \right| < \varepsilon \\ & \text{ iff } \left| \frac{3x+18}{4(2-x)} \right| < \varepsilon \\ & \text{ iff } \frac{3}{4} \frac{|x+6|}{|x-2|} < \varepsilon. \end{aligned}$$

We must "eliminate" the term $|x-2|$.

Assume that $\delta \leq 1$. Then $-7 < x < -5$

$$\begin{array}{c} \delta \quad \delta \\ \text{---} \text{---} \\ -7 \quad x = -6 \quad -5 \end{array}$$

and $7 < |x-2| < 9$ so that

$$\frac{1}{9} < \frac{1}{|x-2|} < \frac{1}{7}. \text{ Thus,}$$

$$\frac{3}{4} \cdot |x+6| \cdot \frac{1}{|x-2|} < \frac{3}{4} |x+6| \cdot \left(\frac{1}{7}\right) < \varepsilon$$

$$\text{iff } \frac{3}{28} |x+6| < \varepsilon$$

$$\text{iff } |x+6| < \frac{28}{3} \varepsilon. \text{ Choose } \delta = \min \left\{ 1, \frac{28}{3} \varepsilon \right\}.$$

Thus, if $0 < |x+6| < \delta$ then $|f(x) + \frac{1}{4}| < \varepsilon$. This completes the proof.

9.) Prove that $\lim_{x \rightarrow 9} (\sqrt{x} + 2) = 5$:

Let $\varepsilon > 0$ be given. Determine $\delta > 0$ (which depends on ε) so that if $0 < |x - 9| < \delta$ then $|f(x) - 5| < \varepsilon$. Begin with $|f(x) - 5| < \varepsilon$ and solve for $|x - 9|$. Then

$$|f(x) - 5| < \varepsilon \text{ iff } |(\sqrt{x} + 2) - 5| < \varepsilon$$

$$\text{iff } |\sqrt{x} - 3| < \varepsilon$$

$$\text{iff } \left| \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(\sqrt{x} + 3)} \right| < \varepsilon$$

$$\text{iff } \frac{|x - 9|}{|\sqrt{x} + 3|} < \varepsilon .$$

We must eliminate the term $|\sqrt{x} + 3|$.

Assume that $\delta \leq 1$. Then $8 < x < 10$ and

$$\frac{\overbrace{\delta} \quad \overbrace{\delta}}{\underbrace{\quad}_{8 \quad x=9 \quad 10}}$$

$$\sqrt{8} + 3 < |\sqrt{x} + 3| < \sqrt{10} + 3$$

$$\text{so that } \frac{1}{\sqrt{10} + 3} < \frac{1}{|\sqrt{x} + 3|} < \frac{1}{\sqrt{8} + 3} .$$

Thus,

$$|x - 9| \cdot \frac{1}{|\sqrt{x} + 3|} < |x - 9| \cdot \frac{1}{\sqrt{8} + 3} < \varepsilon$$

$$\text{iff } |x - 9| < (\sqrt{8} + 3) \varepsilon .$$

Choose $\delta = \min \{1, (\sqrt{8} + 3) \varepsilon\}$. Thus, if

$0 < |x - 9| < \delta$ then $|f(x) - 5| < \varepsilon$. This completes the proof.