# Section 10.1

2) $a_n = \frac{1}{n!}$ so $a_1 = \frac{1}{1!} = 1$, $a_2 = \frac{1}{2!} = \frac{1}{2}$, $a_3 = \frac{1}{3!} = \frac{1}{6}$, $a_4 = \frac{1}{4!} = \frac{1}{24}$

3) $a_n = \frac{(-1)^{n+1}}{2n-1}$ so $a_1 = \frac{(-1)^2}{1} = 1$,

$a_2 = \frac{(-1)^3}{3} = \frac{-1}{3}$, $a_3 = \frac{(-1)^4}{5} = \frac{1}{5}$,

$a_4 = \frac{(-1)^5}{7} = \frac{-1}{7}$

4) $a_n = 2 + (-1)^n$ so $a_1 = 2 + (-1) = 1$,

$a_2 = 2 + (-1)^2 = 2+1 = 3$,

$a_3 = 2 + (-1)^3 = 2 - 1 = 1$,

$a_4 = 2 + (-1)^4 = 2+1 = 3$.

8) $a_1 = 1$, $a_{n+1} = \frac{a_n}{n+1}$ so

$a_2 = \frac{a_1}{2} = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{6} = \frac{1}{3!}$,

$a_4 = \frac{a_3}{4} = \frac{1}{24} = \frac{1}{4!}$, $a_5 = \frac{a_4}{5} = \frac{1}{5!}$,

$a_6 = \frac{1}{6!}$, $a_7 = \frac{1}{7!}$, $a_8 = \frac{1}{8!}$, $a_9 = \frac{1}{9!}$,

$a_{10} = \frac{1}{10!}$

11) $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ so

$a_3 = a_2 + a_1 = 1 + 1 = 2$,

$a_4 = a_3 + a_2 = 2+1 = 3$,

$a_5 = a_4 + a_3 = 3 + 2 = 5$,

$a_6 = a_5 + a_4 = 5 + 3 = 8$,

$a_7 = a_6 + a_5 = 8 + 5 = 13$,
$a_8 = a_7 + a_6 = 13 + 8 = 21$,
$a_9 = a_8 + a_7 = 21 + 13 = 34$,
$a_{10} = a_9 + a_8$

12) $a_1 = 2$, $a_2 = -1$, $a_{n+2} = \frac{a_{n+1}}{a_n}$ so

$a_3 = \frac{a_2}{a_1} = \frac{-1}{2}$,

$a_4 = \frac{a_3}{a_2} = \frac{-1/2}{-1} = \frac{1}{2}$,

$a_5 = \frac{a_4}{a_3} = \frac{1/2}{-1/2} = -1$,

$a_6 = \frac{a_5}{a_4} = \frac{-1}{1/2} = -2$

$a_7 = \frac{a_6}{a_5} = \frac{-2}{-1} = 2$,

$a_8 = \frac{a_7}{a_6} = \frac{2}{-2} = -1$,

$a_9 = \frac{a_8}{a_7} = \frac{-1}{2}$,

$a_{10} = \frac{a_9}{a_8} = \frac{-1/2}{-1} = \frac{1}{2}$

13) $(-1)^{n+1}$ for $n = 1, 2, 3, \ldots$

16) $(-1)^{n+1} \cdot \frac{1}{n^2}$ for $n = 1, 2, 3, \ldots$

20) $-4 + n$ for $n = 1, 2, 3, \ldots$

26) $\lfloor \frac{n}{2} \rfloor$ for $n = 1, 2, 3, \ldots$

28) $-1 \le (-1)^n \le +1 \to$

$n-1 \le n+(-1)^n \le n+1 \to$

$\frac{n-1}{n} \le \frac{n+(-1)^n}{n} \le \frac{n+1}{n}$ ; then

$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right) = 1-0 = 1$ and

$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1+0 = 1$ so

by Sandwich Theorem $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = 1$.

32) $\lim_{n\to\infty} \frac{n+3}{n^2+5n+6} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{2n+5} = \text{"}\frac{1}{\infty}\text{"} = 0$

33) $\lim_{n\to\infty} \frac{n^2-2n+1}{n-1} = \lim_{n\to\infty} \frac{(n-1)^2}{n-1}$

$= \lim_{n\to\infty} (n-1) = \infty$ (diverges)

35) $a_n = 1+(-1)^n : 0, 2, 0, 2, 0, 2, \ldots$ so

$\lim_{n\to\infty} a_n$ DNE (by oscillation)

39) $-1 \le (-1)^{n+1} \le +1 \longrightarrow$

$\frac{-1}{2n-1} \le \frac{(-1)^{n+1}}{2n-1} \le \frac{1}{2n-1}$ ; then

$\lim_{n\to\infty} \frac{-1}{2n-1} = \text{"}\frac{-1}{\infty}\text{"} = 0$ and

$\lim_{n\to\infty} \frac{1}{2n-1} = \text{"}\frac{1}{\infty}\text{"} = 0$ so by Sandwich Theorem $\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1} = 0$.

40) $\lim_{n\to\infty} \left(\frac{-1}{2}\right)^n = 0$ since $-1 < \frac{-1}{2} < 1$

42) $\lim_{n\to\infty} \frac{1}{(0.9)^n} = \lim_{n\to\infty} \frac{1}{\frac{9^n}{10^n}}$

$= \lim_{n\to\infty} \frac{10^n}{9^n} = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = \infty$ (diverges) since $\frac{10}{9} > 1$.

43) $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin\left(\frac{\pi}{2} + 0\right)$

$= \sin\frac{\pi}{2} = 1$

46) $-1 \le \sin n \le +1 \rightarrow 0 \le \sin^2 n \le 1 \rightarrow$

$\frac{0}{2^n} \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$; then

$\lim_{n\to\infty} \frac{0}{2^n} = \lim_{n\to\infty} 0 = 0$ and

$\lim_{n\to\infty} \frac{1}{2^n} = \text{"}\frac{1}{\infty}\text{"} = 0$ so by Sandwich Theorem $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$

48) $\lim_{n\to\infty} \frac{3^n}{n^3} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{3^n \cdot \ln 3}{3n^2} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=}$

$$\lim_{n\to\infty} \frac{3^n \cdot (\ln 3)^2}{6n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{3^n \cdot (\ln 3)^3}{6} = \infty$$

(diverges)

49) $$\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1} = \text{"}\frac{1}{\infty}\text{"} = 0$$

52) $$\lim_{n\to\infty} (0.03)^{1/n} = (0.03)^0 = 1$$

54) $$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{(-1)}{n}\right)^n = e^{-1}$$

60) $$\lim_{n\to\infty} (\ln n - \ln(n+1)) \overset{\text{"}\infty-\infty\text{"}}{=} \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)$$

$$= \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \ln\left(\lim_{n\to\infty} \frac{1}{1}\right) = \ln 1 = 0$$

by continuity of $y = \ln x$

63) $$a_n = \frac{n!}{n^n}$$

$n$: 1, 2, 3, 4, 5, ...

$$\frac{n!}{n^n}: \frac{1}{1}, \frac{2\cdot 1}{2\cdot 2}, \frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3}, \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 4\cdot 4\cdot 4}, \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 5\cdot 5\cdot 5\cdot 5}, \ldots$$

$$= 1, \frac{1}{2}, \left(\frac{2}{3}\right)\left(\frac{1}{3}\right), \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right), \left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{5}\right), \ldots$$

$$\leq 1, \frac{1}{2}, \left(\frac{1}{3}\right)^2, \left(\frac{1}{4}\right)^3, \left(\frac{1}{5}\right)^4, \ldots$$

$$\leq 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \text{ i.e.,}$$

$$0 \le \frac{n!}{n^n} \le \frac{1}{n} \quad ; \quad \text{then}$$

$$\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n} \quad \text{so by}$$

Sandwich Theorem $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

70) $$\lim_{n\to\infty} \frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n}$$

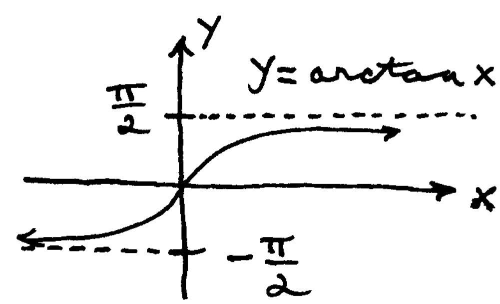
$$= \lim_{n\to\infty} \frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n} \cdot \frac{\frac{1}{(11/12)^n}}{\frac{1}{\left(\frac{11}{12}\right)^n}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{120}{121}\right)^n}{\left(\frac{108}{110}\right)^n + 1} = \frac{0}{0+1} = 0$$

78) $$\lim_{n\to\infty} n\left(1-\cos\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1-\cos\frac{1}{n}}{\frac{1}{n}}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n\to\infty} \frac{\sin\frac{1}{n} \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = \sin 0 = 0$$

81) $$\lim_{n\to\infty} \arctan n = \frac{\pi}{2}$$




87) $$\lim_{n\to\infty} \left(n - \sqrt{n^2-n}\right) = \text{"}\infty - \infty\text{"}$$

$$= \lim_{n\to\infty} \left(n-\sqrt{n^2-n}\right)\frac{\left(n+\sqrt{n^2-n}\right)}{\left(n+\sqrt{n^2-n}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n^2\left(1-\frac{1}{n}\right)}}$$

$$= \lim_{n\to\infty} \frac{n}{n+n\sqrt{1-\frac{1}{n}}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}}$$

$$= \frac{1}{1+\sqrt{1-0}} = \frac{1}{2}$$

99) $x_1 = 1$, $x_{n+1} = x_1 + x_2 + x_3 + \cdots + x_n$, then

$x_2 = x_1 = 1$,

$x_3 = x_1 + x_2 = 1+1 = 2$,

$x_4 = x_1 + x_2 + x_3 = 1+1+2 = 4$,

$x_5 = x_1 + x_2 + x_3 + x_4 = 1+1+2+4 = 8$,

$x_6 = x_1 + x_2 + x_3 + x_4 + x_5 = 1+1+2+4+8 = 16$,

$x_7 = \ldots = 32$, $x_8 = 64$, ...

$x_n = 2^{n-2}$ for $n = 2, 3, 4, 5, \ldots$

108) Prove that $\lim_{n\to\infty} x^{1/n} = 1$ (for $x>1$):

Let $\varepsilon > 0$ be given. Find integer $N$ so that if $n > N$, then $|x^{1/n} - 1| < \varepsilon$.

Then $|x^{1/n} - 1| < \varepsilon$ iff $x^{1/n} - 1 < \varepsilon$ (since $x>1$)

iff $x^{1/n} < \varepsilon + 1$

iff $\ln x^{1/n} < \ln(\varepsilon+1)$

iff $\frac{1}{n} \ln x < \ln(\varepsilon + 1)$

iff $n > \frac{\ln x}{\ln(\varepsilon+1)}$. Let $N$ be any integer $\geq \frac{\ln x}{\ln(\varepsilon+1)}$. Thus, if $n > N$, then $|x^{1/n} - 1| < \varepsilon$.

QED

# Worksheet 1

1.) a.) Prove that $\lim_{n\to\infty} \frac{1}{n+5} = 0$:

Let $\varepsilon > 0$ be given. Find an integer $N$ so that if $n > N$, then $\left|\frac{1}{n+5} - 0\right| < \varepsilon$. Then

$\left|\frac{1}{n+5} - 0\right| < \varepsilon$ iff $\frac{1}{n+5} < \varepsilon$ (assume $n > 0$)

iff $n + 5 > \frac{1}{\varepsilon}$

iff $n > \frac{1}{\varepsilon} - 5$. Choose any integer $N \geq \frac{1}{\varepsilon} - 5$. Thus, if $n > N$, then $\left|\frac{1}{n+5} - 0\right| < \varepsilon$.

QED

b.) Prove that $\lim_{n\to\infty} \frac{3}{\sqrt{n+2}} = 0$:

Let $\varepsilon > 0$ be given. Find an integer $N$ so that if $n > N$, then $\left|\frac{3}{\sqrt{n+2}} - 0\right| < \varepsilon$. Then

$\left|\frac{3}{\sqrt{n+2}} - 0\right| < \varepsilon$ iff $\frac{3}{\sqrt{n+2}} < \varepsilon$

iff $\sqrt{n+2} > \frac{3}{\varepsilon}$

iff $n + 2 > \frac{9}{\varepsilon^2}$

iff $n > \frac{9}{\varepsilon^2} - 2$. Choose any integer $N \geq \frac{9}{\varepsilon^2} - 2$. Thus, if $n > N$, then $\left| \frac{3}{\sqrt{n+2}} - 0 \right| < \varepsilon$.

QED

c.) Prove that $\lim_{n \to \infty} \frac{n+3}{1-n} = -1$ :

Let $\varepsilon > 0$ be given. Find integer $N$ so that if $n > N$, then $\left| \frac{n+3}{1-n} - (-1) \right| < \varepsilon$. Then

$\left| \frac{n+3}{1-n} - (-1) \right| < \varepsilon$ iff $\left| \frac{n+3}{1-n} + \frac{1-n}{1-n} \right| < \varepsilon$

iff $\left| \frac{4}{1-n} \right| < \varepsilon$

iff $\frac{4}{-(1-n)} < \varepsilon$ (assume $n > 1$.)

iff $\frac{4}{n-1} < \varepsilon$

iff $n - 1 > \frac{4}{\varepsilon}$

iff $n > \frac{4}{\varepsilon} + 1$. Choose any integer $N \geq \frac{4}{\varepsilon} + 1$. Thus, if $n > N$, then $\left| \frac{n+3}{1-n} - (-1) \right| < \varepsilon$.

QED