Section 10.10

2) $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3$

$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$

7) $(1+x^3)^{-1/2} = 1 + \frac{-1}{2}(x^3) + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}(x^3)^2 + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(x^3)^3$

$= 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \longrightarrow$

15) $\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$

$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \longrightarrow$

$\int_0^{0.2} \sin(x^2)\,dx = \int_0^{0.2} \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right) dx$

$= \left(\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\right)\Big|_0^{0.2}$

$= \frac{(0.2)^3}{3} - \underbrace{\frac{(0.2)^7}{7\cdot 3!}}_{\approx 3\times 10^{-14} < 10^{-3}} + \frac{(0.2)^{11}}{11\cdot 5!} - \frac{(0.2)^{15}}{15\cdot 7!} + \cdots$

, so by convergent alternating series facts,

$\int_0^{0.2} \sin(x^2)\,dx \approx \frac{(0.2)^3}{3} \approx 0.00267$

with absolute error

$|R_1| \le \frac{(0.2)^7}{7\cdot 3!} \approx 3\times 10^{-14} < 10^{-3}$.

17) $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\,dx = \int_0^{0.1} (1+x^4)^{-1/2}\,dx$

$= \int_0^{0.1} \left[1 + \frac{-1}{2}(x^4) + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}(x^4)^2 + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(x^4)^3 + \cdots\right] dx$

$= \int_0^{0.1} \left[1 - \frac{1}{2}x^4 + \frac{3}{8}x^8 - \frac{5}{16}x^{12} + \cdots\right] dx$

$= \left(x - \frac{1}{10}x^5 + \frac{3}{72}x^9 - \cdots\right)\Big|_0^{0.1}$

$= (0.1) - \frac{1}{10}(0.1)^5 + \frac{3}{72}(0.1)^9 - \cdots$

$(0.1)^6 = 0.000001 \le 0.001 = 10^{-3}$

so by convergent alternating series facts

$\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\,dx \approx 0.1$ with absolute error

$|R_1| \le (0.1)^6 = 0.000001 \le 10^{-3}$

20) $\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1} \left(1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots\right) dx$

$= \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) dx$

$= \left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right)\Big|_0^{0.1}$

$= (0.1) - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5\cdot 2!} - \cdots$

$\approx 0.000333 \le 0.001 = 10^{-3}$

so by convergent alternating series facts

$\int_0^{0.1} e^{-x^2}\,dx \approx 0.1$ with absolute error

$|R_1| \le \frac{(0.1)^3}{3} \approx 0.000333 \le 0.001 = 10^{-3}$

25) $F(x) = \int_0^x \sin t^2 \, dt$

$= \int_0^x \left((t^2) - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} - \frac{(t^2)^7}{7!} + \cdots \right) dt$

$= \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \cdots \right) dt$

$= \left(\frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} - \cdots \right) \Big|_0^x$

$= \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!}$ for $0 \le x \le 1$

$\frac{x^{11}}{11 \cdot 5!} \le \frac{1^{11}}{11 \cdot 5!} \approx 0.00075 \le 10^{-3}$

so by convergent alternating series facts

$F(x) \approx \frac{x^3}{3} - \frac{x^7}{42}$ with absolute error

$|R_2| \le \frac{x^{11}}{11 \cdot 5!} \le 0.00075 \le 10^{-3}$.

28) $F(x) = \int_0^x \frac{\ln(1+t)}{t} \, dt$

$= \int_0^x \frac{1}{t} \cdot \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \right) dt$

$= \int_0^x \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \cdots \right) dt$

$= \left(t - \frac{t^2}{4} + \frac{t^3}{9} - \frac{t^4}{16} \right) \Big|_0^x$

$= x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n^2} + \cdots$

a.) If $0 \le x \le 0.5$, then

$\frac{x^6}{6^2} \le \frac{(0.5)^6}{6^2} \approx 0.000434 \le 10^{-3}$, so

by convergent alternating series facts

$$F(x) \approx x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25}$$

with absolute error

$$|R_5| \le \frac{x^6}{36} \le \frac{(0.5)^6}{36} \approx 0.000434 \le 10^{-3}$$

b.) If $0 \le x \le 1$, then

$$\frac{x^{32}}{32^2} \le \frac{1^{32}}{1024} = \frac{1}{1024} \le \frac{1}{1000} = 10^{-3}, \text{ so}$$

by convergent alternating series facts

$$F(x) \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots + \frac{x^{31}}{31^2} - \frac{x^{32}}{32^2}$$

with absolute error

$$|R_{32}| \le \frac{x^{32}}{32^2} \le \frac{1^{32}}{32^2} = \frac{1}{1024} \le \frac{1}{1000} = 10^{-3}$$

29) a.) $$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{e^x - 1}{2x}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

b.) $$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$$

$$= \lim_{x \to 0} \frac{\left(\cancel{1} + \cancel{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \cancel{(1+x)}}{x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots\right)}{\cancel{x^2}}$$

$$= \frac{1}{2} + 0 + 0 + 0 + \cdots = 1/2$$

32) a.) $\lim_{\theta\to 0} \dfrac{\sin\theta - \theta + \frac{\theta^3}{6}}{\theta^5}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta\to 0} \dfrac{\cos\theta - 1 + \frac{\theta^2}{2}}{5\theta^4} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta\to 0} \dfrac{-\sin\theta + \theta}{20\theta^3}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta\to 0} \dfrac{-\cos\theta + 1}{60\theta^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta\to 0} \dfrac{\sin\theta}{120\theta}$

$= \dfrac{1}{120} \cdot \lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = \dfrac{1}{120} \cdot 1 = \dfrac{1}{120}$

b.) $\lim_{\theta\to 0} \dfrac{\sin\theta - \theta + \theta^3/6}{\theta^5}$

$= \lim_{\theta\to 0} \dfrac{\left(\cancel{\theta} - \cancel{\frac{\theta^3}{3!}} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) - \cancel{\theta} + \cancel{\frac{\theta^3}{3!}}}{\theta^5}$

$= \lim_{\theta\to 0} \dfrac{\cancel{\theta^5}\left(\frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \cdots\right)}{\cancel{\theta^5}}$

$= \dfrac{1}{5!} - 0 + 0 - 0 + 0 - \cdots = \dfrac{1}{120}$

34) a.) $\lim_{Y\to 0} \dfrac{\arctan Y - \sin Y}{Y^3 \cos Y}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{Y\to 0} \dfrac{\frac{1}{1+Y^2} - \cos Y}{-Y^3 \sin Y + 3Y^2 \cos Y}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{Y\to 0} \dfrac{\frac{-2Y}{(1+Y^2)^2} + \sin Y}{-Y^3\cos Y - 3Y^2 \sin Y - 3Y^2 \sin Y + 6Y\cos Y}$

$$= \lim_{Y \to 0} \frac{X \cdot \left[\frac{-2}{(1+Y^2)^2} + \frac{\sin Y}{Y} \right]}{\cancel{X} \cdot \left[-Y^2 \cos Y - 3Y \sin Y - 3Y \sin Y + 6 \cos Y \right]}$$

$$= \frac{-\frac{2}{1^2} + 1}{-0-0-0+6\cos 0} = \frac{-1}{6}$$

6.) $\lim_{Y \to 0} \dfrac{\arctan Y - \sin Y}{y^3 \cos Y}$

$$= \lim_{Y \to 0} \frac{\left(\cancel{Y} - \frac{Y^3}{3} + \frac{Y^5}{5} - \frac{Y^7}{7} + \cdots\right) - \left(\cancel{Y} - \frac{Y^3}{3!} + \frac{Y^5}{5!} - \frac{Y^7}{7!} + \cdots\right)}{Y^3 \left(1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - \frac{Y^6}{6!} + \cdots\right)}$$

$$= \lim_{Y \to 0} \frac{\frac{-1}{6} Y^3 + \frac{23}{120} Y^5 - \frac{719}{5040} Y^7 + \cdots}{Y^3 \left(1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - \frac{Y^6}{6!} + \cdots\right)}$$

$$= \lim_{Y \to 0} \frac{\cancel{Y^3} \cdot \left(\frac{-1}{6} + \frac{23}{126} Y^2 - \frac{719}{5040} Y^4 + \cdots\right)}{\cancel{Y^3} \cdot \left(1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - \frac{Y^6}{6!} + \cdots\right)}$$

$$= \frac{-1/6 + 0 - 0 + 0 - 0 + \cdots}{1 - 0 + 0 - 0 + 0 - \cdots} = -1/6$$

59) a.) $\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}} \longrightarrow$

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\, dx = \int \left(1 + (-x^2)\right)^{-1/2} dx$$

$$= \int \left[1 + \frac{-1}{2}(-x^2) + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}(-x^2)^2 + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(-x^2)^3 + \cdots \right] dx$$

$$= \int \left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots \right] dx$$

$$= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$$

62) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \longrightarrow$

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + (x^2)^4 + \cdots \longrightarrow$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots \xrightarrow{D}$$

$$\left(D\left(\frac{1}{1-x^2}\right) = \frac{(1-x^2)(0) - (1)\cdot(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}. \right)$$

$$\frac{2x}{(1-x^2)^2} = 2x + 4x^3 + 6x^5 + 8x^7 + \cdots$$

$$= \sum_{n=1}^{\infty} 2n \cdot x^{2n-1}$$