Section 14.4

1.) $w = x^2 + 2y$, $x = \cos t$, $y = \sin t$

I.) $\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_y \cdot \frac{dy}{dt}$

$= (2x) \cdot (-\sin t) + (2) \cdot (\cos t)$

$= (2\cos t)(-\sin t) + 2\cos t$

$= 2\cos t\,(1 - \sin t)$

OR

II.) $w = x^2 + 2y = (\cos t)^2 + 2(\sin t) \xrightarrow{D}$

$\frac{dw}{dt} = 2(\cos t) \cdot (-\sin t) + 2\cos t$

$= 2\cos t\,(1 - \sin t)$

if $t = \pi$, then $\frac{dw}{dt} = 2\cos\pi\,(1 - \sin\pi)$

$= 2(-1)(1-0) = -2$

6.) $w = z - \sin(xy)$, $x = t$, $y = \ln t$, $z = e^{t-1}$

I.) $\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_y \cdot \frac{dy}{dt} + w_z \cdot \frac{dz}{dt}$

$= -\cos(xy) \cdot y \cdot (1) + -\cos(xy) \cdot x \cdot \left(\frac{1}{t}\right) + (1) \cdot e^{t-1}$

$= -\cos(t\ln t) \cdot \ln t - \cos(t\ln t) \cdot \cancel{t}\left(\frac{1}{\cancel{t}}\right) + e^{t-1}$

$= -\cos(t\ln t) \cdot (\ln t + 1) + e^{t-1}$

OR

II.) $w = z - \sin(xy) = e^{t-1} - \sin(t\ln t) \xrightarrow{D}$

$\frac{dw}{dt} = e^{t-1} - \cos(t\ln t) \cdot \left[t \cdot \frac{1}{t} + (1)\ln t\right]$

$= e^{t-1} - \cos(t\ln t) \cdot [1 + \ln t]$

if $t=1$, then $\frac{dw}{dt} = e^{0} - \cos(\overset{0}{\cancel{\sin 1}})\cdot[1+\overset{0}{\cancel{\sin 1}}]$

$= 1 - 1(1) = 0$

8.) $z = \arctan\left(\frac{x}{y}\right)$, $x = u\cos v$, $y = u\sin v$

I.) $\frac{\partial z}{\partial u} = z_x \cdot \frac{\partial x}{\partial u} + z_y \cdot \frac{\partial y}{\partial u}$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}\cdot\cos v + \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{-x}{y^2}\cdot\sin v$$

$$= \frac{y}{y^2+x^2}\cdot\cos v + \frac{-x}{y^2+x^2}\cdot\sin v$$

$$= \frac{u\sin v\cdot\cos v}{u^2\sin^2 v+u^2\cos^2 v} + \frac{-u\cos v\cdot\sin v}{u^2\sin^2 v+u^2\cos^2 v}$$

$= 0$ OR

II.) $z = \arctan\left(\frac{x}{y}\right) = \arctan\left(\frac{\cancel{u}\cos v}{\cancel{u}\sin v}\right) \rightarrow$

$z = \arctan(\cot v) \xrightarrow{D}$

$\frac{\partial z}{\partial u} = 0$ (since v is constant);

if $(u,v) = (1.3, \pi/6)$, then $\frac{\partial z}{\partial u} = 0$;

I.) $\frac{\partial z}{\partial v} = z_x \cdot \frac{\partial x}{\partial v} + z_y \cdot \frac{\partial y}{\partial v}$

$$= \frac{y}{y^2+x^2}\cdot -u\sin v + \frac{-x}{y^2+x^2}\cdot u\cos v$$

$$= \frac{u\sin v\cdot(-u\sin v)}{u^2\sin^2 v+u^2\cos^2 v} + \frac{-u\cos v\cdot(u\cos v)}{u^2\sin^2 v+u^2\cos^2 v}$$

$$= \frac{-u^2(\sin^2 v+\cos^2 v)}{u^2(\sin^2 v+\cos^2 v)} = -1 \qquad \text{OR}$$

II.) $z = \arctan\left(\frac{x}{y}\right) = \arctan(\cot v) \xrightarrow{D}$

$$\frac{\partial z}{\partial v} = \frac{1}{1+(\cot v)^2} \cdot -\csc^2 v = \frac{-\csc^2 v}{\csc^2 v} = -1 \; ;$$

if $(u,v) = (1.3, 6)$, then $\frac{\partial z}{\partial v} = -1$.

9.) $w = xy + yz + xz$, $x = u+v$, $y = u-v$, $z = uv$

I.) $$\frac{\partial w}{\partial u} = w_x \cdot \frac{\partial x}{\partial u} + w_y \cdot \frac{\partial y}{\partial u} + w_z \cdot \frac{\partial z}{\partial u}$$

$$= (y+z)\cdot(1) + (x+z)\cdot(1) + (x+y)\cdot v$$

$$= (u-\cancel{v}) + uv + (u+\cancel{v}) + uv + ((u+v)+(u-v))\cdot v$$

$$= 2u + 2uv + 2uv = 2u + 4uv$$

OR

II.) $$w = xy + yz + xz$$

$$= (u+v)(u-v) + (u-v)(uv) + (u+v)(uv)$$

$$= u^2 - v^2 + u^2 v - \cancel{uv^2} + u^2 v + uv^2$$

$$= u^2 - v^2 + 2u^2 v \xrightarrow{D}$$

$\frac{\partial w}{\partial u} = 2u + 4uv$; if $(u,v) = (\frac{1}{2}, 1)$, then $\frac{\partial w}{\partial u} = 2(\frac{1}{2}) + 4(\frac{1}{2})(1) = 1 + 2 = 3$;

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I.) $$\frac{\partial w}{\partial v} = w_x \cdot \frac{\partial x}{\partial v} + w_y \cdot \frac{\partial y}{\partial v} + w_z \cdot \frac{\partial z}{\partial v}$$

$$= (y+z)(1) + (x+z)(-1) + (x+y)(u)$$

$$= (u-v) + uv + ((u+v)+uv)(-1) + ((u+\cancel{v})+(u-\cancel{v}))(u)$$

$$= \cancel{u} - v + \cancel{uv} - \cancel{u} - v - \cancel{uv} + 2u^2$$

$$= 2u^2 - 2v$$

OR

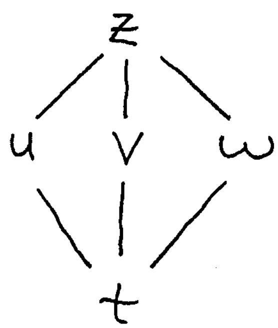
II.) $w = xy + yz + xz = u^2 - v^2 + 2u^2 v \xrightarrow{D}$

$$\frac{\partial w}{\partial v} = -2v + 2u^2 = 2u^2 - 2v \, ;$$

if $(u,v) = \left(\frac{1}{2}, 1\right)$, ~~then~~

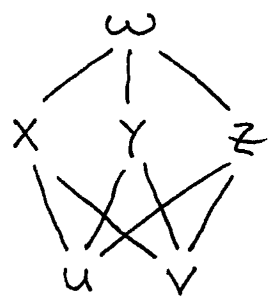
$$\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -2 + \frac{1}{2} = -\frac{3}{2} .$$

14.)



$$\frac{dz}{dt} = f_u \cdot \frac{du}{dt} + f_v \cdot \frac{dv}{dt} + f_w \cdot \frac{dw}{dt}$$

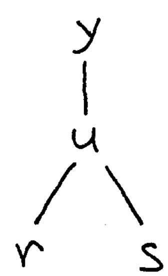
15.)



$$\frac{\partial w}{\partial u} = w_x \cdot \frac{\partial x}{\partial u} + w_y \cdot \frac{\partial y}{\partial u} + w_z \cdot \frac{\partial z}{\partial u}$$

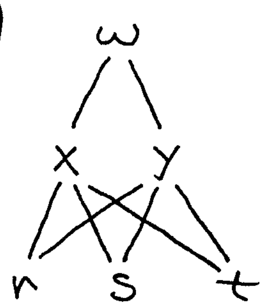
$$\frac{\partial w}{\partial v} = w_x \cdot \frac{\partial x}{\partial v} + w_y \cdot \frac{\partial y}{\partial v} + w_z \cdot \frac{\partial z}{\partial v}$$

20.)



$$\frac{\partial y}{\partial r} = \frac{dy}{du} \cdot \frac{\partial u}{\partial r}$$

24.)



$$\frac{\partial w}{\partial s} = w_x \cdot \frac{\partial x}{\partial s} + w_y \cdot \frac{\partial y}{\partial s}$$

26.) $\underbrace{xy + y^2 - 3x - 3}_{F(x,y)} = 0$;

By Theorem 8, $\frac{dy}{dx} = \frac{-F_x}{F_y}$ $\longrightarrow$

$$\frac{dy}{dx} = \frac{-(y-3)}{x+2y}$$ (Let $(x, y) = (-1, 1)$.) $\longrightarrow$

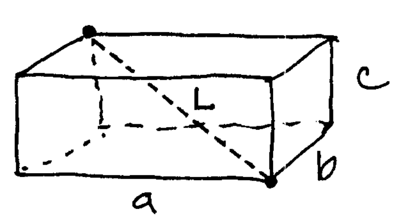
$$\frac{dy}{dx} = \frac{-(1-3)}{-1+2(1)} = \frac{2}{1} = 2$$

28.) $\underbrace{xe^y + \sin xy + y - \ln 2}_{F(x,y)} = 0$;

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(e^y + \cos xy \cdot y)}{xe^y + \cos xy \cdot x + 1}$$

(Let $(x, y) = (0, \ln 2)$.) $\longrightarrow$

$$\frac{dy}{dx} = -\frac{(e^{\ln 2} + \overset{1}{\cancel{\cos}}\, 0 \cdot \ln 2)}{0 + 0 + 1} = -2 - \ln 2$$

42) 

Given $\frac{da}{dt} = 1$ m./sec.,

$\frac{db}{dt} = 1$ m./sec., and $\frac{dc}{dt} = -3$ m./sec.

when $a = 1$ m., $b = 2$ m., and $c = 3$ m.

volume $V = abc$; surface area $S = 2ab + 2bc + 2ac$; diagonal $L = \sqrt{a^2+b^2+c^2}$;

a.) Find $\frac{dV}{dt}$: (Use triple product rule.)

$$\frac{dV}{dt} = \frac{da}{dt}\cdot(bc) + \frac{db}{dt}(ac) + \frac{dc}{dt}(ab)$$

$$= (1)(2\cdot 3) + (1)(1\cdot 3) + (-3)(1\cdot 2)$$

$$= 6 + 3 - 6 = +3 \text{ m.}^3/\text{sec.}$$

b.) Find $\frac{dS}{dt}$:

$$\frac{dS}{dt} = 2\left[\left(a\cdot\frac{db}{dt} + \frac{da}{dt}\cdot b\right) + \left(b\cdot\frac{dc}{dt} + \frac{db}{dt}\cdot c\right) + \left(a\cdot\frac{dc}{dt} + \frac{da}{dt}\cdot c\right)\right]$$

$$= 2\left[(1\cdot 1 + 1\cdot 2) + (2\cdot(-3) + 1\cdot 3) + (1\cdot(-3) + 1\cdot 3)\right]$$

$$= 2\left[3 + (-3) + (0)\right] = 2(0) = 0 \text{ m.}^2/\text{sec.}$$

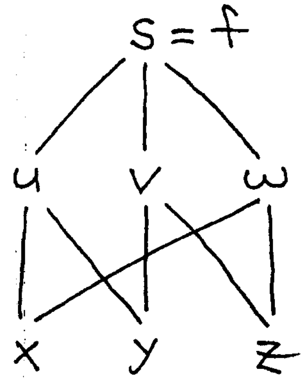
c.) Find $\frac{dL}{dt}$:

$$\frac{dL}{dt} = \cancel{\frac{1}{2}}(a^2+b^2+c^2)^{-1/2}\cdot\left[\cancel{2}a\frac{da}{dt} + \cancel{2}b\cdot\frac{db}{dt} + \cancel{2}c\cdot\frac{dc}{dt}\right]$$

$$= \frac{1}{\sqrt{14}}\left[1\cdot 1 + 2\cdot 1 + 3\cdot(-3)\right] = \frac{-6}{\sqrt{14}} \text{ m./sec.}$$

(The diagonal is $\downarrow$.)

43) assume function $s = f(u, v, w)$ and $u = x - y$, $v = y - z$, $w = z - x$.



Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0 :$$

Using the chain rule →

$$\frac{\partial f}{\partial x} = s_u \cdot \frac{\partial u}{\partial x} + s_w \cdot \frac{\partial w}{\partial x}$$
$$= s_u \cdot (1) + s_w \cdot (-1) = s_u - s_w \ ;$$

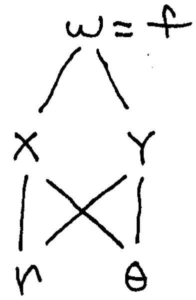
$$\frac{\partial f}{\partial y} = s_u \cdot \frac{\partial u}{\partial y} + s_v \cdot \frac{\partial v}{\partial y}$$
$$= s_u \cdot (-1) + s_v \cdot (1) = s_v - s_u \ ;$$

$$\frac{\partial f}{\partial z} = s_v \cdot \frac{\partial v}{\partial z} + s_w \cdot \frac{\partial w}{\partial z}$$
$$= s_v \cdot (-1) + s_w \cdot (1) = s_w - s_v \ ;$$

then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = (\cancel{s_u} - \cancel{s_w}) + (\cancel{s_v} - \cancel{s_u}) + (\cancel{s_w} - \cancel{s_v}) = 0 .$$

44) assume function $w = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$. Then



a.) $\frac{\partial w}{\partial r} = f_x \cdot \frac{\partial x}{\partial r} + f_y \cdot \frac{\partial y}{\partial r}$

$= f_x \cdot (\cos\theta) + f_y \cdot (\sin\theta)$;

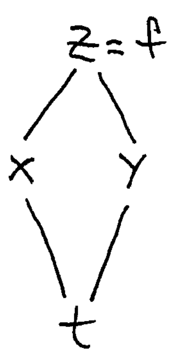
$\frac{\partial w}{\partial \theta} = f_x \cdot \frac{\partial x}{\partial \theta} + f_y \cdot \frac{\partial y}{\partial \theta}$

$= f_x \cdot (-r\sin\theta) + f_y \cdot (r\cos\theta)$

$= r(-f_x \cdot \sin\theta + f_y \cdot \cos\theta) \longrightarrow$

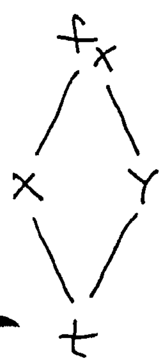
$\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \cdot \sin\theta + f_y \cdot \cos\theta$.

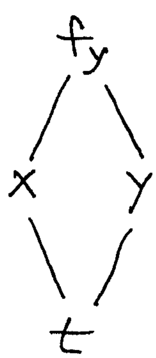
1.) a.) assume $z = f(x, y)$ and $x = e^{2t}$, $y = \sin t$. Then by the chain rule



$\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$

$= f_x \cdot 2e^{2t} + f_y \cdot \cos t$; and





$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right)$

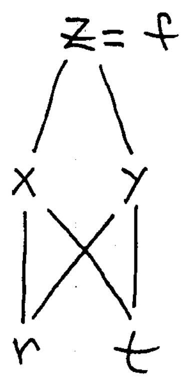
$= \frac{d}{dt}\left[f_x \cdot 2e^{2t} + f_y \cdot \cos t\right]$

$= f_x \cdot \frac{d}{dt}(2e^{2t}) + \frac{d}{dt}(f_x) \cdot 2e^{2t}$

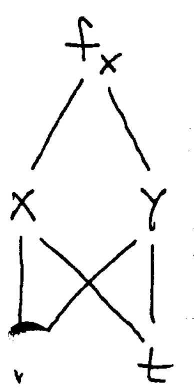
$+ f_y \cdot \frac{d}{dt}(\cos t) + \frac{d}{dt}(f_y) \cdot \cos t$

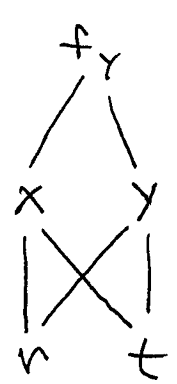
$$= f_x \cdot 4e^{2t} + \left[f_{xx} \cdot \frac{dx}{dt} + f_{xy} \cdot \frac{dy}{dt} \right] \cdot 2e^{2t}$$
$$+ f_y \cdot (-\sin t) + \left[f_{yx} \cdot \frac{dx}{dt} + f_{yy} \cdot \frac{dy}{dt} \right] \cdot \cos t$$
$$= f_x \cdot 4e^{2t} + \left[f_{xx} \cdot 2e^{2t} + f_{xy} \cdot \cos t \right] \cdot 2e^{2t}$$
$$- f_y \cdot \sin t + \left[f_{xy} \cdot 2e^{2t} + f_{yy} \cdot \cos t \right] \cdot \cos t$$
$$= f_x \cdot (4e^{2t}) - f_y \cdot (\sin t)$$
$$+ f_{xx} \cdot (4e^{4t}) + f_{yy} \cdot (\cos^2 t)$$
$$+ f_{xy} \cdot (4e^{2t} \cos t) \quad .$$

1.) 6.) assume $z = f(x,y)$ and $x = rt^2$, $y = r^3 - t$. Then by chain rule



i.) $$\frac{\partial z}{\partial t} = f_x \cdot \frac{\partial x}{\partial t} + f_y \cdot \frac{\partial y}{\partial t}$$
$$= f_x \cdot (2rt) + f_y \cdot (-1)$$
$$= f_x \cdot (2rt) - f_y$$; then





$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\partial z}{\partial t} \right]$$
$$= \frac{\partial}{\partial t} \left[f_x \cdot (2rt) - f_y \right]$$
$$= f_x \cdot \frac{\partial}{\partial t}(2rt) + \frac{\partial}{\partial t}(f_x) \cdot (2rt)$$
$$- \frac{\partial}{\partial t}(f_y)$$

$$= f_X \cdot 2r + \left[f_{XX} \cdot \frac{\partial X}{\partial t} + f_{XY} \cdot \frac{\partial Y}{\partial t} \right] \cdot (2rt)$$
$$- \left[f_{YX} \cdot \frac{\partial X}{\partial t} + f_{YY} \cdot \frac{\partial Y}{\partial t} \right]$$
$$= f_X \cdot 2r + f_{XX} \cdot (2rt)(2rt) + f_{XY} \cdot (-1)(2rt)$$
$$- f_{XY} \cdot (2rt) - f_{YY} \cdot (-1)$$
$$= f_X \cdot (2r) + f_{XX} \cdot (4r^2t^2)$$
$$- f_{XY} \cdot (4rt) + f_{YY}$$

ii.) $$\frac{\partial z}{\partial r} = f_X \cdot \frac{\partial X}{\partial r} + f_Y \cdot \frac{\partial Y}{\partial r}$$
$$= f_X \cdot t^2 + f_Y \cdot 3r^2$$; then
$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial r} \right] = \frac{\partial}{\partial r}\left[f_X \cdot t^2 + f_Y \cdot 3r^2 \right]$$
$$= \frac{\partial}{\partial r}[f_X] \cdot t^2 + f_Y \cdot \frac{\partial}{\partial r}(3r^2) + \frac{\partial}{\partial r}(f_Y) \cdot 3r^2$$
$$= \left[f_{XX} \cdot \frac{\partial X}{\partial r} + f_{XY} \cdot \frac{\partial Y}{\partial r} \right] \cdot t^2$$
$$+ f_Y \cdot 6r + \left[f_{YX} \cdot \frac{\partial X}{\partial r} + f_{YY} \cdot \frac{\partial Y}{\partial r} \right] \cdot 3r^2$$
$$= f_{XX} \cdot (t^2)(t^2) + f_{XY}\,(3r^2)(t^2)$$
$$+ f_Y \cdot 6r + f_{XY} \cdot (t^2)(3r^2) + f_{YY} \cdot (3r^2)(3r^2)$$
$$= f_{XX} \cdot (t^4) + f_Y \cdot (6r) + f_{YY} \cdot (9r^4)$$
$$+ f_{XY} \cdot (6r^2t^2)$$